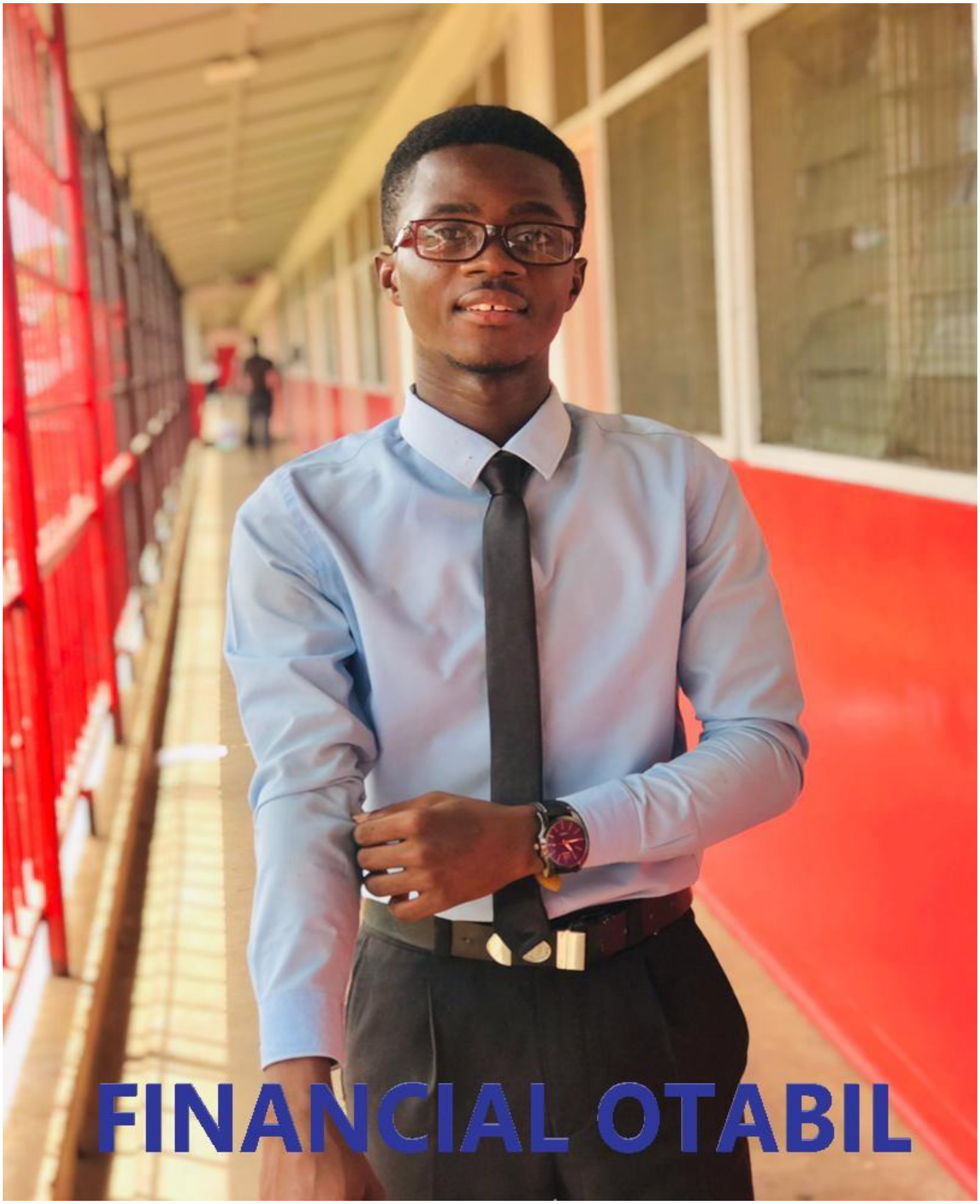

For all your slides, past questions and assignments, visit my website at: www.augustinedotcom.wordpress.com

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QUANTITATIVE METHODS

Course Title	Quantitative Methods
Course Code	ISD 251
Program (E.g. BBA or MBA)/Level	BBA 2
Core or Elective	Core
Instructor(s) Name and email address	Emmanuel Quansah (equansah@hotmail.com)
Number of Class sessions in course	
Duration of each class	2 Hrs
Credit Hours	3
Semester	1

COURSE OVERVIEW

Managers make decisions in complex circumstances and for this they need many skills, including problem solving, leadership, communications, analysis, reasoning, experience and judgement. Many of their decisions are based on numerical information. For instance, they have to consider income, profit, production levels, productivity, interest rates, forecast demand, costs and all the other information that is presented as numbers. And this means that managers must have some understanding of quantitative methods. This does not mean that managers have to be professional mathematicians, but they do need to understand quantitative reasoning and be able to interpret numerical results. The aim of this course is to familiarize the student with the quantitative aspects of managerial decision making. This course provides the student with the concepts, methods and tools for the application of logical and quantitative analysis to business decision making and problem solving. The course highlights the benefits as well as the limits of quantitative analysis in a real-world context. It familiarizes the student with a number of important mathematical concepts underpinning decision making and problem analysis, including basics of algebra, calculus, matrices, probability, forecasting and simulation.

Because management students come from different backgrounds, the course starts with basic introductory concepts. The course then works from basic principles and develops ideas in a logical sequence, moving from underlying concepts through to real applications. This course is has a practical rather than a theoretical approach.

COURSE OBJECTIVES

Upon completion of this course a student should be able to:

1. Appreciate the importance and benefit of numbers
2. Understand the process of decision-making
3. Apply mathematical concepts to the decision-making process
4. Solve quantitative-based business problems
5. Develop an ability to use computer softwares/programs in solving quantitative-based business problems

COURSE CONTENT

Module 1: Elementary Algebra

1. Properties of numbers
2. Surds
3. Simple Algebraic Expressions (simplification, factorisation, fractions, change of subject)
4. Indices
5. Logarithms
6. Functions
7. Polynomial and Quadratic Equations
8. Linear and Simultaneous Equations
9. Set Notation

Module 2: Calculus - Differentiation

1. Introduction to differentiation
2. Rules of differentiation
3. Implicit Differentiation
4. Maxima and Minima
5. Applications of differentiation

Module 3: Calculus - Integration

1. The indefinite Integral
2. The definite integral
3. Applications of Integration

Module 4: Matrices

1. Matrix Operations
2. Determinants of a Matrix
3. Cramer's Rule
4. Applications of Matrices

Module 5: Forecasting

1. Introduction to Forecasting
2. Qualitative Models
3. Quantitative Models (Moving averages, weighted moving averages and exponential smoothing)
4. Measures of forecast accuracy (MAD, MSE and MAPE)

Module 6: Probability Distribution

1. Combinations and Permutations
2. Binomial Distribution
3. Poisson Distribution
4. **Normal Distribution**

ASSESSMENT/GRADING

The criteria for assessing students shall include continuous assessments made up of; assignments, mid-semester examination, class attendance and students' participation in class as well as end of semester examination. Course grades will be as follows;

Continuous Assessment	30%
Quiz 1	5
Homework/Individual Assignments	5
Group Assignment	10
Mid-sem	10
Final Exam	70%
Total	100%

COURSE MATERIAL

Teaching notes and course hand-outs used during the lectures will be provided to the students at no cost.

Recommended Text:

1. N.D. Vohra, *Quantitative Techniques in Management* , (Tata Mcgraw-Hill)
2. Barry Render, Ralph M. Jr. Michael E. Hanna, *Quantitative Analysis for Management*, Pearson
3. Waters, D., (2011), "Quantitative Methods for Business", Pearson, Essex.
4. Mohammed, (2003), "*Quantitative Methods for Business Economics*", Prentice Hall of India, New Delhi.

FINANCIAL OTABIL

ISD 251 QUANTITATIVE METHODS

LECTURE MATERIAL

S. K. AMPONSAH

J. ANNAN

TABLE OF CONTENT

CHAPTER ONE

DIFFERENTIATION

INTRODUCTION

In this unit, we are going to talk about differentiation, where we shall concentrate on the standard results, rules, differentiation of exponential and logarithmic functions, as well as parametric, maxima and minima, test points, and then the applications of differentiation. The unit will end with implicit differentiation.

NOTATION

Differentiation is the process of finding the derivative of a function. The derivative of a function is also called its derived function and also its derived coefficient.

Rules of Differentiation

1.0 If $y = x^n$ then

$$\frac{dy}{dx} = nx^{n-1}$$

Examples (i) If $y = x^3$, then $\frac{dy}{dx} = 3x^2$

(ii) If $y = x^5$ then

$$\frac{dy}{dx} = 5x^4$$

Note: If $y = au$ where a is a constant and u is a function of x,

then $$\frac{dy}{dx} = a\frac{du}{dx}$$

Example

If $y = 7x^5$

$$\frac{dy}{dx} = 7(5x^4) = 35x^4$$

1.1 DIFFERENTIATION OF SUMS AND DIFFERENCES

Here, we differentiate term by term.

Example

If $y = x^3 + 2x$

$$\frac{dy}{dx} = 3x^2 + 2$$

dx

1.2 DIFFERENTIATION OF A CONSTANT

If $y = 5$, find $\frac{dy}{dx}$.

Solution $y = 5$,

can be written as $y = 5x^0$

$$\frac{dy}{dx} = (0)(5)x^{-1}$$

$$= 0$$

Note: Differentiation of a constant is zero

Example

If $y = x^3 - 2x + 1000$, find $\frac{dy}{dx}$.

$$\frac{dy}{dx} = 3x^2 - 2$$

dx

1.3 Differentiation of Exponential Functions

If $y = e^{f(x)}$ then $\frac{dy}{dx} = f'(x)e^{f(x)}$

Example

1. If $y = e^{5x+3}$ then $\frac{dy}{dx} = 5e^{5x+3}$

2. If $y = 5e^{4x^2+2x+8}$ then $\frac{dy}{dx} = 5(8x + 2)e^{4x^2+2x+8}$

3. If $y = e^x$ then $\frac{dy}{dx} = 1.e^x = e^x$

1.4 Differentiation of Logarithmic Functions

If $y = \ln[f(x)]$, then $\frac{dy}{dx} = \frac{f'(x)}{f(x)}$

Examples

dy 6x+4

1. If $y = \ln x(3x^2 + 4x + 5)$, then $\frac{dy}{dx} = \frac{\quad}{3x^2 + 4x + 5}$

2. If $y = \ln(3 - x^2)$, then $\frac{dy}{dx} = \frac{-2x}{3 - x^2}$

3. If $y = \ln x(5x - 2)$, then $\frac{dy}{dx} = \frac{5}{5x - 2}$

4. If $y = \ln x$, then $\frac{dy}{dx} = \frac{1}{x}$

1.5 DIFFERENTIATION OF PRODUCT OF TWO FUNCTIONS

If y = If u and v are both functions of x and $y = uv$ then

$$\frac{dy}{dx} = v.\frac{du}{dx} + u.\frac{dv}{dx}$$

Examples

(i) If y□ $(x^2$ □ $2)(2x$□ $4)$

Let u□x^2 □2 and v□$2x$□4

du dv

___□ $2x$ and ___□ 2

Then dx dx

dy du dv

___ □ v. □u. ___ ^{2} 2)2

Using dx dx dx = $(2x$□$4)2x$□$(x$ □

= $4x^2$ □$8x$□$2x^2$ □4

= $6x^2$ □$8x$□4

^{2} ln$(3x^2$□ □$8x$ 4), —

find dy. ( )*ii If y* □ *x dx Solution*

Hereu □ x^2, *and v* □ ln$(3x^2$□ □$8x$ 4) *du dv*

$6x$□8

___dx □ $2x$ *and* dx □ $(3x^2$□ □$8x$ 4)

dy du dv □ v □u

dx dx dx

$\frac{dy}{dx}$ □ $_2$ $8x$

$4)2x$□ x_2 $(3x_2 6$□ □$x 8$□$x 8\ 4)\ \ln(3x$ □ □

$^{(3x□2)}\ln(3$ □ $x^2),$find $dy.$ (*iii*)

If y □ e

Solution

dx *and* v □

Here, u □ $e^{(3x□2)}$

$\ln(3$ □ $x^2)$

du

$^{(3x□2)}$ □

and $dx dv$ □ 3□□$2xx_2$

$3e$

dx

dy du dv □ v □

u

dx dx dx

$2x$

dy

dx □ $\ln(3$ □ $x_2)3e^{(3x□2)}$ □ $e^{(3x□2)}(3$□□

$x_2)$ □ $3e^{(3x□2)}\ln(3$ □ $x_2)$ □ $e^{(3x□2)}$

$(3$□□$2x^{x}{}_2)$

$$= e^{(3x+2)}\left\{3\ln(3+x^2) + \frac{2x}{(3+x^2)}\right\}$$

1.6 QUOTIENT RULE

If u and v are both functions of x and $y = \dfrac{u}{v}$ then,

$$\frac{dy}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$$

Example

If $y = \dfrac{2x+1}{x^2}$

Solution

Let $u = 2x+1$ and $v = x^2$

du dv

Then $\dfrac{du}{dx} = 2$ and $\dfrac{dv}{dx} = 2x$

$$\frac{dy}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$$

Using

$$\frac{x^2.(2) - (2x-1).2x}{x^4} = \frac{2x^2 - (4x^2 - 2x)}{x^4}$$

$$= \frac{-2x^2 + 2x}{x^4}$$

$$= \frac{-2x(x-1)}{x^4}$$

$$= \frac{-2(x-1)}{x^3}$$

1.7 Higher Derivatives

If the derivative of a function of x is differentiated with respect to x, the 2nd derivative of the function is obtained. If the 2nd derivative is differentiated, the 3rd derivative is obtained, and so on. The 2nd, 3rd,…, nth derivatives of y with respect to x are usually written as $\frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \ldots, \frac{d^ny}{dx^n}$ respectively. The usual function notation is $f'(x)$, $f''(x)$,...., $f^{(n)}(x)$.

Example

If $y = x^6 - 4x^2 - \frac{3}{x}$,

$$\frac{dy}{dx} = 6x^5 - 8x + \frac{3}{x^2},$$

$$\frac{d^2y}{dx^2} = 30x^4 - 8 - \frac{6}{x^3},$$

$$\frac{d^3y}{dx^3} = 120x^3 + \frac{18}{x^4}$$

$3 \qquad x$

1.8 Chain Rule

If y is a function of u and u is a function of x, then y is called a function of x. This can be differentiated using the chain rule, $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$. It is useful to remember that, by the chain rule

$\frac{d}{dx}(y^2) = 2y\frac{dy}{dx}$, $\frac{d}{dx}(y^3) = 3y^2\frac{dy}{dx}$ and so on.

Example

Differentiate $y = (3x^4 - 5)^7$ with respect to x

Let $u = 3x^4 - 5$ then $y = u^7$

so $\frac{du}{dx} = 12x^3$ and $\frac{dy}{du} = 7u^6$

using $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 7u^6 \cdot 12x^3 = 84x^3(3x^4 - 5)^6$

1.9 IMPLICIT DIFFERENTIATION

If $y = x^2 - 4x + 2$, y is completely defined in terms of x, therefore, y is called an explicit function of x. Where the relationship between x and y

is more involved, it may not be possible (or desirable) to separate *y* completely on the left-hand side, e.g. $xy + y^2 = 2$. In such a case as this, *y* is called an implicit function of x, because a relationship of the form $y = f(x)$ is implied in the given equation.

It may still be necessary to determine the differential coefficient of *y* with respect to *x* and in fact is not all difficult. All we have to remember is that *y* is a function of x, even if it is difficult to see that $x^2 + y^2 = 25$ is an example of an implicit function. Once again, all we have to remember is that *y* is a function of x.

So if $x^2 + y^2 = 25$, let us find $\frac{dy}{dx}$.

Differentiating with respect to *x*, we obtain

$$2x + 2y\frac{dy}{dx} = 0$$

$$y\frac{dy}{dx} = -x$$

$$\frac{dy}{dx} = -\frac{x}{y}$$

	Note **To Differentiate an implicit function** a) Differentiate it term by term to give an equation in x, y and $\frac{dy}{dx}$ Make $\frac{dy}{dx}$ the subject of the equation *To obtain the second derivative* a) Differentiate the $\frac{dy}{dx}$, equation to obtain an equation for $\frac{d^2y}{dx}$ b) Substitute for $\frac{dy}{dx}$ if necessary.

Example 1

If $x^2 + y^2 - 2x - 6y + 5 = 0$, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at $x = 3$, $y = 2$

Solution

Differentiate as it stands with respect to x.

$dy \qquad dy$

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$$2x \square 2y \frac{dy}{dx} \square 2 \square 6 \frac{dy}{dx} \square 0$$

$$\therefore (2y \square 6)\frac{dy}{dx} \square 2 \square 2x$$

$$\frac{dy}{dx} \square \frac{2 \square 2x}{2y \square 6}$$

$\therefore$ At (3, 2),

$$\frac{dy}{dx} \square \frac{2 \square 2(3)}{2(2) \square 6}$$

$$\square \frac{2 \square 6}{4 \square 6} = \frac{\square 4}{\square 2} = 2$$

Then

$$\frac{d^2y}{dx^2} \square \frac{d}{dx}\left(\frac{1 \square x}{y \square 3}\right)$$

$$\square \frac{(y \square 3)(\square 1) \square (1 \square x)\frac{dy}{dx}}{(y \square 3)^2}$$

$$\frac{d^2y}{dx^2} \quad (2\square \square 3)(1) - (1\text{-}3)^2 \quad 1\ (4) \square \square$$

At (3,2) $\frac{d^2y}{dx^2} = \frac{\qquad}{(2-3)^2} = \frac{\qquad}{1} = 5$

Example 3

If $x^2 + 2xy + 3y^2 = 4$, find $\frac{dy}{dx}$

Solution

Differentiating term by term, we have

$$2x + 2\left(x\frac{dy}{dx} + y\right) + 6y\frac{dy}{dx} = 0$$

$$2x + 2x\frac{dy}{dx} + 2y + 6y\frac{dy}{dx} = 0$$

$$(2x + 6y)\frac{dy}{dx} + 2x + 2y = 0$$

$$(2x+6y)\frac{dy}{dx} = -2x-2y$$

$$\frac{dy}{dx} = \frac{-(x+y)}{2x+6y}$$

Example 4

If $x^3 + y^3 = 3xy^2 + 8$, find $\frac{dy}{dx}$

Solution

Differentiating term by term, we have

$$3x^2 + 3y^2\frac{dy}{dx} = 3\left(x.2y\frac{dy}{dx} + y^2.1\right) + 0$$

$$3x^2 + 3y^2\frac{dy}{dx} = 6xy\frac{dy}{dx} + 3y^2 + 0$$

$$(3y^2 - 6xy)\frac{dy}{dx} = -3x^2 + 3y^2$$

$$\frac{dy}{dx} = \frac{-3x^2 + 3y^2}{3y^2 - 6xy}$$

Example 5

Given that $x^2 - 3xy + 2y^2 - 2x = 4$, find the value of $\frac{dy}{dx}$ at the point (1,-1).

Solution

Differentiating with respect to x, we have

$$2x - 3\left(x\frac{dy}{dx} + y\right) + 4y\frac{dy}{dx} - 2 = 0$$

$$2x - 3x\frac{dy}{dx} - 3y + 4y\frac{dy}{dx} - 2 = 0$$

$$(-3x + 4y)\frac{dy}{dx} = 2 + 3y - 2x$$

$$\frac{dy}{dx} = \frac{2 + 3y - 2x}{-3x + 4y}$$

at (1,-1) we have

$$\frac{dy}{dx} = \frac{2 - 3 - 2}{-3 - 4} = \frac{-3}{-7} = \frac{3}{7}$$

Example 6

If $x^2 + 2y^2 = 10$ find (i) $\frac{dy}{dx}$ (ii) $\frac{d^2y}{dx^2}$

Solution

i) Differentiating term by term, we have

$$2x + 4y\frac{dy}{dx} = 0$$

$$4y\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = \frac{-2x}{4y} = \frac{-x}{2y}$$

ii) Differentiating $\frac{dy}{dx}$, with respect to x, we have

$$\frac{2y(-1) - (-x)2\frac{dy}{dx}}{(2y)^2}$$

$$= \frac{-2y + 2x\left(\frac{-x}{2y}\right)}{(2y)^2}$$

$$= \frac{-2y - \frac{x^2}{y}}{(2y)^2} = \frac{-2y^2 - x^2}{4y^3}$$

	EXERCISES Find (i) $\frac{dy}{dx}$ if: i. $x^3 + y^3 = 3xy$ ii. $3x^2 + 4xy + y^2 - 6x = 10$ iii. $3x^2y^2 + 3xy + 4y^2 + 10x = 2$ iv. $x^2y \square y^2x \square 3xy^3 \square 6x \square 3$

1.10 Logarithmic Differentiation

To differentiate a function of the form $y = [f(x)]^{g(x)}$

(a) Take logarithms of the given function

(b) Differentiate the new function as usual

Example 1

If $y = x^{2x}$ find $\frac{dy}{dx}$.

Solution

$y = x^{2x}$ taking logs on both sides, we have $\ln y = 2x\ln x$

Differentiating, we have $\frac{1}{y}\frac{dy}{dx} = 2x \cdot \frac{1}{x} + 2\ln x$

$$\frac{dy}{dx} = y(2 + 2\ln x) = x^{2x}(2 + 2\ln x)$$

Example 2

If $y = 3^{x^2}$ find $\frac{dy}{dx}$

$\ln y = x^2 \ln 3$

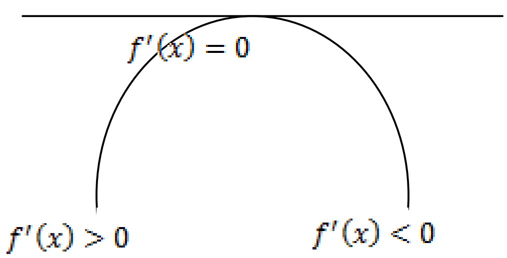
$\frac{1}{y}\cdot\frac{dy}{dx} = 2x\ln 3$

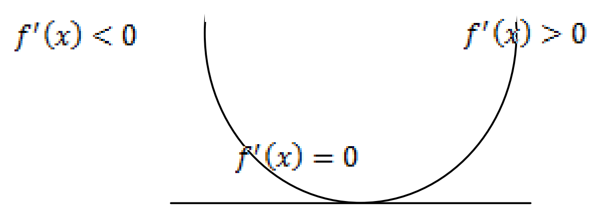
$$\frac{dy}{dx} = y(2x\ln 3) = 3^{x^2}(2x\ln 3)$$

1.11 Maxima and Minima

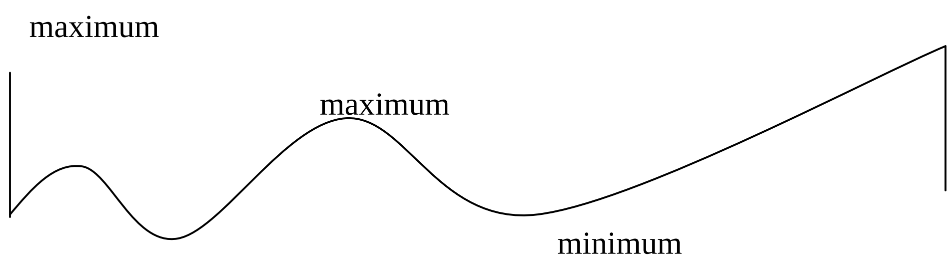
At a point of local maximum, a function has a greater Value than at points immediately on either side of it. At a point of local minimum, a function has a smaller Value than at points immediately on either side of it. Local maxima and minima are also called turning points.

A function may have more than one turning point. The local maxima and minima are not necessarily the greater or least Values of the function in





Local



the given range.

Local greatest

1.12 Tests for Points

A stationary point is a point at which $f'(x) = 0$. Local maxima, minima and horizontal points of inflexion are stationary points. To test for stationary points,

a) Find $f'(x) = 0$ and $f''(x) = 0$

b) Put $f'(x) = 0$ and solve the resulting equation to find the x – coordinate(s) of the point(s)

c) Find $f''(x) = 0$ at the stationary point(s).

i.) if $f''(x) < 0$, the point is local maximum ii.)

if $f''(x) > 0$, the point is local minimum

iii.) if $f''(x) = 0$, find the sign of $f'(x)$ for a value of x just to the left and just to the right of the point.

Sign to the Left	Sign to the Right	Types of point
+	-	Maximum
-	+	Minimum
+ -	+ -	}point of inflexion

To test general points of inflexion.

a) Find $f''(x)$

b) Put $f''(x) = 0$ and solve the resulting equation to find the possible x– coordinate(s)

c) Find the sign of $f''(x)$ for a value of x just to the left and to the right of the point. If $f''(x)$ changes sign, the point is a point of inflexion.

Example 1

Find the stationary points of $f(x) = \frac{1}{3}x^3 - 2x^2 + 3x$ and identify their nature.

Solution

$$f(x) = \frac{1}{3}x^3 - 2x^2 + 3x$$

$$f'(x) = x^2 - 4x + 3$$

S$f''(x) = 2x - 4$

At stationary points $f'(x) = 0$,

i.e., $x^2 - 4x + 3 = 0$,

$(x - 3)(x - 1) = 0$

$$x = 3 \text{ and } x = 1$$

When $x = 3$, $f''(3) = 2(3) - 4 = 2 > 0$

$$f(3) = \frac{1}{3}(3)^3 - 2(3)^2 + 3(3) = 0$$

Therefore (3, 0) is a local minimum.

When $x = 1$, $f''(1) = 2(1) - 4 = -2 < 0$

$$f(1) = \frac{1}{3}(1)^3 - 2(1)^2 + 3(1) = \frac{4}{3}$$

Therefore (1, $\frac{4}{3}$) is a local maximum.

Example 2

Find any points of inflexion of $f(x) = \frac{1}{3}x^3 - 2x + 3$.

Solution

$$f(x) = \frac{1}{3}x^3 - 2x^2 + 3x$$

$$f'(x) = x^2 - 4x$$

$$f''(x) = 2x - 4$$

At a general point of inflexion $f''(x) = 0$, i.e., $2x - 4 = 0 \Rightarrow x = 2$

For $x = 2^+$, $f''(x) > 0$ i.e. $f''(x)$ changes sign

For $x = 2^-$, $f''(x) < 0$

So $x = 2$ is a general point of inflexion.

1.13 Applications of Maxima and Minima

Maxima and Minima can be applied to practical problems in which the maximum or minimum value of a quantity is required. The procedure is

a) Write an expression for the required quantity.

b) Use the given conditions to rewrite it in terms of a single variable.

c) Find the turning point(s) and their type(s). It is often obvious from the problem itself whether a maximum or minimum has been obtained.

COST, REVENUE AND PROFIT FUNCTIONS

1.14 MARGINAL COST

In business and economics one is often interested in the rate at which something is taking place. A manufacturer, for example, is not only interested in the total cost $C(x)$ at certain production levels x, but also interested in the rate of change of costs at various production levels.

In economics the word marginal refers to a rate of change; that is, to a derivative. Thus, if

$\mathrm{C}(x) = \text{Total cost of production of } x \text{ units during some unit of time}$

$\mathrm{C}'(x) = \text{Marginal Cost}$
$= \text{rate of change in cost per unit chang in production at an output level of } x \text{ uni}$

The marginal cost indicates the change in cost for a unit change in production at a production level of x units if the rate were to remain constant for the next unit change in production.

Example 1

Suppose the total cost (x) in thousands of cedis for manufacturing x bags of cement per week is given by

$C(x) = 2 + 8x - x^2 \quad 0 \leq x \leq 4$

Find

(i) The marginal cost at x

(ii) The marginal cost at $x = 1, 2,$ and 3 levels of production.

Solution

(i) $C'(x) = 8 - 2x$

(ii) $C'(1) = 8 - 2(1) = 6$ ₵6,000 per unit increase in production

(iii) $C'(2) = 8 - 2(2) = 4$ ₵4,000 per unit increase in production

(iv) $C'(3) = 8 - 2(3) = 2$ ₵2,000 per unit increase in production

Notice that, as production goes up, the marginal cost goes down, as we might expect.

Example 2

The total cost per day, $C(x)(in\ hundreds\ of\ cedis)$ for manufacturing *x* tones of steel is given by $C(x) = 3 + 10x - x^2 \quad 0 \leq x \leq 5$

(i) Find the marginal cost at *x*

(ii) Find the marginal cost at *x* = 1, 3, and 4 units level of production.

Marginal Analysis in Business and Economics

- Marginal cost, Revenue, and Profit
- Applications
- Marginal Average cost, Revenue and Profit

Marginal Cost, Revenue, and Profit

One important use of calculus in business and economics is in marginal analysis. Economists also talk about marginal revenues and marginal profit.

If x is the number of units of product produced in some time interval then,

Total cost = C(x)

Marginal cost = C‟(x)

Total revenue = R(x)

Marginal revenue = R‟(x)

Total Profit = $P(x) = R(x) - C(x)$

Marginal Profit = $P'(x) = R'(x) - C'(x)$

= (marginal revenue) - (marginal cost)

- The marginal cost approximates the change in total cost that results from a unit change in production.
- Since C(x) is the total cost of providing (x + 1) units, the change in the total cost

$$(cost\ for\ x+1\ unit) - (cost\ for\ x\ units)$$

$\Delta C = C\ (x+1) - C\ (x)$ is also the cost of producing the (x+1)st item. Thus, the marginal cost C‟(x) also approximates the cost of producing the (x + 1)st item.

If $\Delta x = 1$, then

$\Delta C = C(x+1) - C(x)$

= Exact change in total cost per unit change in production at a production level of *x* units.

Example 3

A small machine shop manufactures drill bits used in the petroleum industry. The shop manager estimates that the total daily cost in cedis of producing *x* bits is $C(x) = 1000 + 25x - \frac{x^2}{10}$,

Find

(i) $C'(10)$ and interpret your result.

(ii) $C'(10) - C'(11)$ and interpret your result.

Solution

$$C'(x) = 25 - \frac{x}{5}$$

$$C'(10) = 25 - \frac{10}{5}$$

$$= 25 - 2$$

$$= 23$$

At production level of 10 bits, a unit increase in production will increase the total production cost by approximately ₵23. Also the cost of producing the 11th bit is approximately ₵23

A STRATEGY FOR SOLVING APPLIED OPTIMIZATION PROBLEMS

Step 1: Introduce variables and construct a mathematical model of the form

Maximize (or Minimize) f(x) on the interval I

Step 2: Find the absolute maximum (or minimum) value of f(x) on the interval I and the value(s) of x where this occurs.

Step 3: Use the solution to the mathematical model to answer the questions asked in the application.

Example 4

A company manufactures and sells x transistor radios per week. If the weekly cost and price- demand equations are:

$$C(x) = 5000 + 2x$$

$$P = 10 - \frac{x}{1000}, \quad 0 \le x \le 8000$$

Find for each week

(i) The maximum revenue

(ii) The maximum profit,

(iii) the production level that will realize the maximum profit

(iv) the price that the company should charge for each radio.

Solution

(i) The revenue received for selling *x* radios at ₵p per radio is

$$R(x) = xP$$

$$= x(10 - \frac{x}{1000})$$

$$= 10x - \frac{x^2}{1000}$$

Thus the mathematical model is

$$Maximize\ R(x) = 10x - \frac{x^2}{1000}, 0 \leq x \leq 8000$$

$$\frac{dR}{dx} = 10 - \frac{x}{500}$$

At the critical value, $\frac{dR}{dx} = 0$

$$\gg\ 10 - \frac{x}{500} = 0$$

$$\gg\ x = 5000$$

Use the second derivative test for absolute extrema.

$$\frac{d^2R}{dx^2} = -\frac{1}{500} < 0\ for\ all\ x$$

Thus, the maximum revenue is

$$Max\ R(x) = R(5000) = 10(5000) - \frac{(5000)^2}{1000}$$

₵25000

Profit = Revenue – Cost

$$P(x) = R(x) - C(x)$$

$$= \left(10x - \frac{x^2}{1000}\right) - (5000 + 2x)$$

$$= 10x - \frac{x^2}{1000} - 5000 - 2x$$

$$= 8x - \frac{x^2}{1000} - 5000$$

The mathematical model is

$$Maximize = P(x) = 8x - \frac{x^2}{1000} - 5000, 0 \leq x \leq 8000$$

$$\frac{dP}{dx} \ 8 - \frac{x}{500}$$

$$8 - \frac{x}{500} = 0$$

$$x = 4000$$

$$\frac{d^2P}{dx^2} = -\frac{1}{500} < 0 \ for \ all \ x.$$

since $x = 400$ is the only critical value and $\frac{d^2P}{dx^2} < 0$,

$$Max\, P(x) = P(4000) = 8(4000) - \frac{(4000)^2}{1000} - 5000$$

$$= ₵11{,}000$$

Now using the price-demand equation with $x = 4000$, we find

$$P = 10 - \frac{4000}{1000} = ¢6$$

Example 5

Repeat Example (4) for

$C(x) = \ 90000 + 30x$

$P = 300 - \frac{x}{300}, 0 \leq x \leq 9000$

Example 6

In example (4) the government has decided to tax the company ₵2 for each radio produced. Taking into consideration this additional cost, how many radios should the company manufacture each in order to maximize its weekly profit?

What is the maximum weekly profit?

How much should it charge for the radios?

Solution

The tax of ₵2 per unit changes the company‟s cost equation:

C(x) = original cost + tax

= 5000 + 2x + 2x

= 5000 + 4x

The new profit function is

$P(x) = R(x) - C(x)$

$$= 10x - \frac{x^2}{1000} - 5000 - 4x$$

$$= 6x - \frac{x^2}{1000} - 5000$$

Thus, we must solve the following

$$maximize P(x) = 6x - \frac{x^2}{1000} - 5000, 0 \leq x \leq 8000$$

$$\frac{dP}{dx} = 6 - \frac{x}{500}$$

$$= 6 - \frac{x}{500}$$

$$x = 3000$$

$$\frac{d^2P}{dx^2} = -\frac{1}{500} < 0 \; for \; all \; x.$$

$$Max\, P(x) = P(3000) = 8(3000) - \frac{(3000)^2}{1000} - 5000$$

$$= ₵4{,}000$$

Using the price-demand equation with x = 3000, we find

$$P = 10 - \frac{3000}{1000} = ₵7$$

Thus the company‟s maximum profit is ₵4000 when 3000 radios are produced and sold weekly at a price of ₵7.

Even though the tax caused the company‟s cost to increase by ₵2 per radio, the price that the company should charge to maximize its profit increases by only ₵1. The company must absorb the other cost ₵1 with a resulting decrease of ₵7000 in maximum profit.

Example 7

A cocoa grower estimates from past records that if twenty trees are planted per acre, each tree with average 60 pounds of nuts per year. If for each additional tree planted per acre (up to fifteen) the average yield per tree drops 2 pounds, how many trees should be planted to maximize the yield per acre?

What is the maximum yield?

Solution

Let *x* be the number of additional trees planted per acre. Then

$20 + x$= Total number of trees planted per acre.

$20 + x$= Yield per tree.

Yield per acre = (Total number of trees per acre)(yield per tree)

$Y(x) = (20 +)(60 - 2x)$

$= 1200 + 20x - 2x^2 , 0 \leq x \leq 15$

Thus, we must solve the following

$$Maximize\, Y(x) = 1200 + 20x - 2x^2$$

$$\frac{dY}{dx}\ 20 - 4x$$

$$20 - 4x = 0$$

$$x = 5\frac{d^2Y}{dx^2} = \ -4 < 0\ for\ all\ x$$

Hence $Max\, Y(x) = Y(5) = 1250$ pounds per acre. Thus, a maximum yield od 1250 pounds of nuts per acre is realized of twenty-five trees are planted per acre.

	EXERCISE Repeat Example, starting with thirty trees per acre and a reduction of 1 pound per tree for each additional tree planted.

CHAPTER TWO

INTEGRATION

THE INDEFINITE INTEGRAL

In our study of the theory of the firm, we have worked with total cost, total revenue and the profit functions and have found their marginal functions. In practice, it is often easier for a company to measure marginal cost, revenue, and profit.

If the marginal revenue of a firm is given by $MR = 300 - 0.5Q$, where Q is the number of units sold

If we want to use this function to find the total revenue function, we need to find $R(Q)$ from the fact that $MR = \frac{dR}{dQ}$. In this situation, we need to reverse the process of differentiation.

This process is called integration. By integration we can write $R(Q) = \int (300 - 0.5Q)\, dQ$

In general $\int x^n\,dx = \frac{x^{n+1}}{n+1} + K$ [Increase the exponent of x by 1 and divide by the new power] and K is an arbitrary constant.

Example 1

Evaluate $\int 3dx$

Solution

$$\int 3dx = \int 3x^0\,dx = \frac{3x^{0+1}}{0+1} + K$$

$$= 3x + K$$

Example 2

Evaluate $\int 8x^5\,dx$

Solution

$$\int 8x^5\,dx = \frac{8}{6}x^{5+1} + K$$

$$= \frac{4}{3}x^6 + K$$

Example 3

Evaluate $\int (3x^2 + 2x + 1)dx$

Solution

$$\int (3x^2 + 2x + 1)dx = \frac{3}{3}x^{2+1} + \frac{2}{2}x^{1+1} + \frac{1}{1}x^{0+1} + K$$

$$= x^3 + x^2 + x + K$$

Example 4

The marginal revenue in dollars per unit for a motherboard is $\overline{MR} = 300 - 0.2x$, where x represent the quantity sold. Find the

(i) revenue function;

(ii) total revenue from the sale of 1000 motherboards.

Solution

(i) We know that the marginal revenue can be found by differentiating the total revenue function. That is, $R'(x) = 300 - 0.2x$

Thus integrating the marginal revenue function gives the total revenue function

$$R(x) = \int (300 - 0.2x)\,dx$$

$$= 300x - \frac{0.2}{2}x^2 + K$$

$$= 300x - 0.1x^2 + K$$

But there is no revenue, when no units are sold, thus $R = 0$, when $x = 0$

$$0 = 300(0) - 0.1(0)^2 + K$$

$$K = 0$$

$\therefore$ The total revenue function is: $R(x) = 300x - 0.1x^2$

(ii) The total revenue from the sale of 1000 motherboards is

$$R(1000) = 300(1000) - 0.1(1000)^2$$
$$= 300{,}000 - 100{,}000$$
$$= \$200{,}000$$

Example 5

Suppose the marginal cost function for a month for a certain product is

$MC = 3Q + 50$, where Q is the number of units and the cost in cedis. If the fixed costs related to the product amount to GH¢100 per month, find the total cost function for the month.

Solution

The total cost function is: $C(Q) = \int (3Q + 50)\,dQ$

$$= \frac{3Q^{1+1}}{1+1} + \frac{50Q^{0+1}}{0+1} + K$$

$$= \frac{3}{2}Q^2 + 50Q + K$$

But when $Q = 0$, $FC = 100$

$$100 = \frac{3}{2}(0)^2 + 50(0) + K$$
$$K = 100$$
$$\therefore C(Q) = \frac{3}{2}Q^2 + 50Q + 100$$

Example 6

A firms marginal cost function for a product is $MC = 2Q + 50$, its marginal revenue function is $MR = 200 - 4Q$ and that the cost of production and sale of 10 units is GH¢700. Find the

(i) optimal level of production;

(ii) profit function

(iii) profit or loss at the optimal level.

Solution

(i) Profit is maximized when $MR = MC$

$$200 - 4Q = 2Q + 50$$

$$200 - 50 = 2Q + 4Q$$

$$150 = 6Q$$

$$25 = Q$$

The level of production that will maximize profit is 25 units

$$R(Q) = \int (200 - 4Q)\,dQ$$

$$= 200Q - \frac{4Q^2}{2} + K$$

$$= 200Q - 2Q^2 + K$$

But there is no revenue, when no units are sold, thus $R = 0$, when $x = 0$

$$0 = 200(0) - 2(0)^2 + K$$

$$K = 0$$

$\therefore$ The total revenue function is: $R(x) = 200Q - 2Q^2$

The total cost function is: $C(Q) = \int (2Q + 50)\,dQ$

$$= \frac{2Q^{1+1}}{1+1} + \frac{50Q^{0+1}}{0+1} + K$$

$$= Q^2 + 50Q + K$$

But when $Q = 10$, $C(Q) = 700$

$$700 = (10)^2 + 50(10) + K$$

$$700 = 100 + 500 + K$$

$$K = 700 - 600$$

$$= 100$$

$$\therefore C(Q) = Q^2 + 50Q + 100$$

(ii) Profit (π) = Revenue − Cost

$$= (200Q - 2Q^2) - (Q^2 + 50Q + 100)$$

$$= 200Q - 2Q^2 - Q^2 - 20Q - 100$$

$$= 180Q - 3Q^2 - 100$$

(iii) $\pi(25) = 180(25) - 3(25)^2 - 100$

$$= 4{,}500 - 1{,}875 - 100$$

$$= \text{GH¢}2{,}525$$

Exercise

1. Evaluate each of the following

(i) $\int (3x^2 + 2)dx$

(ii) $\int (3Q - 5)dQ$

(iii) $\int (6x^2 - 2x + 4)dx$

(iv) $\int (12Q^3 - 15Q^2 + 8Q + 6)dQ$

2. The marginal revenue (in dollars per unit) for a month for a commodity is MR□□$0.01Q$□25, find the total revenue function

3. If the marginal revenue (in cedis per unit) for a month is given by MR□ 450□$0.3Q$, what is the total revenue from the production and sale of 50 units

4. If the monthly marginal cost for a product is MC □ □$2x$ 100, with fixed cost amounting to \$200, find the total cost function for the month.

6. A certain firm‟s marginal cost for a product is MC Q□ □6 60, its marginal revenue is MR□180 2□ Q, and the total cost of producing 10 items is GH¢1000. Find the

(i) optimal level of production;

(ii) profit function;

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[1] . If the marginal cost for a product is MC □ □4 $2x$ and the production of 10 units results in a total cost of \$300, find the

(i) total cost function

(ii) total cost of producing 200 units of the product.

(iii) profit or loss at the optimal level of production.

CHAPTER THREE

BINOMIAL DISTRIBUTION

INTRODUCTION

Some experiments can result in only two possible outcomes. For example the answer to a question may be yes or no. If a coin is tossed, we may obtain either a Head or a Tail. A person selected at random may be male or female; a student may be wearing glasses or not wearing glasses. Thus the result of a trial is one of the two complementary results.

Suppose that the experiment is performed a fixed number of times, *n*, say, and that the probability, *p*, of obtaining one particular outcome (i.e. what we are interested in) called the probability of success, remains the same from trial to trial. The probability, $q = 1 - p$ of the other outcome (i.e. what we are not interested in) is called probability of failure. In this case, we observe that $p + q = 1$. The repeated trials are independent and that the total number of successes is the variable of interest.

An experiment with these characteristics is said to fit the Binomial model and the outcome is a binomial variable.

DEFINITION

Let *E* be an event and *p* be the probability that *E* will happen in any single trial. The number *p* is called the probability of a success. Then $q = 1 - p$ is the probability of a failure. The probability that the even *E* will happen exactly *x* times in *n* trials is given by

$$P(X = x) = {}^{n}C_x P^x q^{n-x}, \quad x = 0,1,2,\ldots,n$$

Here n = the number of trials

P = probability of success

q = probability of failure and

x = the number of successes

Note: The sum of all probabilities $f.e$ from $x = 0$ to $x = n$ must be equal to 1. $\left(i.e.\ P(X \leq n) = 1\right)$.

Illustrative Examples

1. A fair coin is tossed 5 times. Find the probability of obtaining:

i) exactly 2 heads

ii) exactly 1 head

iii) no head

iv) at least one head

v) at most 2 heads

Solution

Here

$$P(head) = \frac{1}{2} = P$$

$$P(tail) = \frac{1}{2} = q$$

$$n = 5$$

Now we can see that from (i) – (v), we are interested in the number of heads and so the probability of head becomes the probability of success.

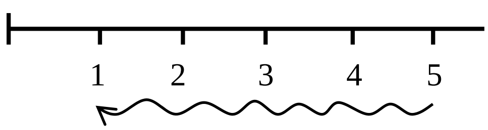
i) $P(\text{exactly 2 heads}) = P(X=2) = {}^{5}C_{2}\left(\frac{1}{2}\right)^{2}\left(\frac{1}{2}\right)^{3} = 10 \times \frac{1}{4} \times \frac{1}{8} = \frac{10}{32} = \frac{5}{16}$

ii) $P(\text{exactly 1 head}) = P(x=1) = {}^{5}C_{1}\left(\frac{1}{2}\right)^{1}\left(\frac{1}{2}\right)^{4} = 5 \times \frac{1}{2} \times \frac{1}{16} = \frac{5}{32}$

iii) $P(\text{no head}) = P(x=0) = {}^{5}C_{0}\left(\frac{1}{2}\right)^{0}\left(\frac{1}{2}\right)^{5} = 1 \times 1 \times \left(\frac{1}{2}\right)^{5} = \frac{1}{32}$

iv) $P(\text{at least one head}) = P(x \geq 1) =$

$P(x=1)+P(x=2)+P(x=3)+P(x=4)+P(x=5)$



This can be evaluated as $P(x \geq 1) = 1 - P(x=0) = 1 - \frac{1}{32} = \frac{31}{32}$

v) $P(\text{at most 2 heads}) = P(x \leq 2) = P(x=0) + P(x=1) + P(x=2)$

$= \frac{1}{32} + \frac{5}{32} + \frac{10}{32} = \frac{16}{32} = \frac{1}{2}$

Example 2

It is known 20% of parts produced by a certain machine are defective. If six parts produced by the machine are selected at randomly from a day"s rum, find the probability that:

i) all six of the parts are defective

ii) none of them is defective iii)

exactly 2 of them are defective

Solution

$$n = 6$$

$$P(\text{defective}) = \frac{20}{100} = 0.2 \quad (P)$$

$$\therefore P(\text{non} - \text{defective}) = 1 - 0.2 = 0.8 \quad (q)$$

Here from (i) to (iii) we are interested in defective parts and so

$$P(\text{defective}) = P(\text{probability of success}).$$

i) $P(\text{all are defective}) = P(x = 6) = {}^{6}C_{6}(0.2)^{6}(0.8)^{0}$

$= 1 \times (0.2)^{6} \times 1 = \underline{0.000064}$

ii) $P(\text{none of them is defective}) = P(x = 0) = {}^{6}C_{0}(0.2)^{0}(0.8)^{6}$

$= \quad 1 - (1 - 0.8)^6 = \quad \underline{0.262144}$

6

iii) $P(\text{exactly 2 are defective}) = P(x = 2) = {}^6C_2 (0.2)^2 (0.8)^4 = \underline{\underline{0.24576}}$

Example 3

A fair die is thrown five times. Calculate, correct to three decimal places, the probability of obtaining

i) at most two sides ii)

exactly three sides

Solution

The set of all possible outcomes when a die is thrown is

$S = \{1,2,3,4,5,6\}$

$\frac{1}{6}$ $P(a\,6) \quad P(\text{not a}$ $\frac{5}{6}$

$6)$

Here we are interested in the number of sides and therefore $p = \frac{1}{6}$ and $q = \frac{5}{6}$ and

$n = 5$.

i) $P(\text{at most two sixes}) = P(x = 0) + P(x = 1) + P(x = 2)$

$$= {}^5C_0\left(\frac{1}{6}\right)^0\left(\frac{5}{6}\right)^5 + {}^5C_1\left(\frac{1}{6}\right)^1\left(\frac{5}{6}\right)^4 + {}^5C_2\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^3$$

= 0.4019 + 0.4019 + 0.1608

= 0.9646

= 0.965 to 3 decimal places

ii) $P(\textit{exactly three sixes}) = P(x = 3)$

$$= {}^5C_3\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^2 = 0.03215$$

= 0.032 to 3 decimal places

Example 4

A machine that manufactures engine parts has an average probability of 0.2 of breaking down. If a factory has 10 of these machines, what is the probability that at least 8 will be in good working order.

Solution

$P(\textit{breakingdown}) = 0.2 = q$

$P(\textit{good workingorder}) = 1 - 0.2 = 0.8 = P$

$n = 10$

We are required to find the probability of being in good working order and therefore p=0.8 and q=0.2

$$P(at least\ 8) = P(x=8) + P(x=9) + P(x=10)$$

$$= {}^{10}C_8 (0.8)^8 (0.2)^2 + {}^{10}C_9 (0.8)^9 (0.2)^2 + {}^{10}C_{10} (0.8)^{10} (0.2)^0$$

$$= 0.3020 + 0.2684 + 0.1074$$

$$= 0.6778$$

Example 5

A box contains 12 balls, three of which are defective. If a random sample of 5 is drawn from the box one after the other with replacement, what is the probability that (a) exactly one is defective?

(b) at most one is defective?

Solution

$$P(defective) = \frac{3}{12} = 0.25 = P$$

$$P(non-defective) = 1 - 25 = 0.75 = q$$

$$n = 5$$

(a) $P(\text{exactly one is defective}) = P(x = 1)$

$$= {}^{5}C_{1}(0.25)^{1}(0.75)^{4} = 0.3955$$

(b) $P(\text{at most one defective}) = P(x = 0) + P(x = 1)$

$$= {}^{5}C_{0}(0.25)^{0}(0.75)^{5} + 0.3955$$

$$= 0.2373 + 0.3955 = \underline{\underline{0.6328}}$$

Example 6

In a certain game of gambling a player tosses a fair coin; if it falls head she wins N100 and if it falls tail she losses N100. A player with N800 tosses the coin six times. What is the probability that she will be left with N600.

Solution

Here $n = 6$

$$P(Head) = \frac{1}{2} = P$$

$$P(Tail) = \frac{1}{2} = q$$

and let x be the number of times the player wins. If the player started with N800 and at the end of six games, she was left with N600, then it means she won only 2 out of the six games.

$$P(x = 2) = {}^{6}C_{2}\left(\frac{1}{2}\right)^{2}\left(\frac{1}{2}\right)^{4} = 15 \times \frac{1}{4} \times \frac{1}{16} = \frac{15}{64} = \underline{\underline{0.2344}}$$

Example 7

In an examination 60% of the candidates passed. Find the probability that a random sample of 15 candidates from this class will include at most 3 failures.

Solution

$P(pass) = 0.6$

$P(fail) = 1 - 0.6 = 0.4$ $n = 15$

Here we are interested in the number of failures and therefore $p = p(fail) = 0.4$ and q=0.6

$$P(atmost\,3\ failures) = P(x = 0) + P(x = 1) + P(x = 2) + P(x = 3)$$

$$= {}^{15}C_{0}(0.4)^{0}(0.6)^{15} + {}^{15}C_{1}(0.4)^{1}(0.6)^{14} + {}^{15}C_{2}(0.4)^{2}(0.6)^{13}$$

$$+ {}^{15}C_{3}(0.4)^{3}(0.6)^{12}$$

$$= 0.00047 + 0.0047 + 0.0219 + 0.0634$$

$$= \underline{\underline{0.28757}}$$

Example 8

A farmer produces seeds in packets for sale. The probability that a seed selected at random will grow is 0.80. If 6 of these seeds are sown, what is the probability that

i) less than two will grow? ii) less

than two will not grow? iii) exactly

half the seeds will grow?

Solution

Number of seeds, $n = 6$

$P(grow) = 0.8$

$P(not\ grow) = 0.2$

i) Here $P = 0.8$, $q = 0.2$, $n = 6$

P(less than two will grow) = P (X=0) + P(X=1)

$$= {}^{6}C_{0}(0.8)^{0}(0.2)^{6} + {}^{6}C_{1}(0.8)^{1}(0.2)^{5}$$

$= \underline{\underline{0.0016}}$

ii) P(less than 2 will <u>not</u> grow)

Here we are interested in not grow

Hence p=0.2 and q=0.8 $P(x < 2) = P(x = 0) + P(x = 1)$

$$= {}^{6}C_{0}(0.2)^{0}(0.8)^{6} + {}^{6}C_{1}(0.2)^{1}(0.8)^{5}$$

$= \underline{\underline{0.65536}}$

iii) P(exactly half of the seeds will grow) $\frac{1}{2} \times 6 = 3$

Thus $P(x = 3)$ where p=0.8, q=0.2 and n=6

$P(x = 3) = {}^{6}C_{3}(0.8)^{3}(0.2)^{3} = \underline{\underline{0.08192}}$

Example 9

A question paper contains 8 multiple-choice questions, each with 4 answers of which only one is the correct answer. If a student guesses at the answers, find the probability that he gets

i) no correct answer ii)

exactly 3 correct answers

iii) at most 3 correct

answers

Solution

$$P(\textit{he answers correctly}) = \frac{1}{4} = P$$

$$P(\textit{he answers wrongly}) = \frac{3}{4} = q$$

i) $P(x=0) = {}^{8}C_{0}\left(\frac{1}{4}\right)^{0}\left(\frac{3}{4}\right)^{8} = \underline{\underline{0.1001129}}$

ii) $P(x=3) = {}^{8}C_{3}\left(\frac{1}{4}\right)^{3}\left(\frac{3}{4}\right)^{5} = \underline{\underline{0.2076}}$

iii) $P(x \le 3) = P(x=0) + P(x=1) + P(x=2) + P(x=3)$

$$= 0.1001 + {}^{8}C_{1}\left(\frac{1}{4}\right)^{1}\left(\frac{3}{4}\right)^{7} + {}^{8}C_{2}\left(\frac{1}{4}\right)^{2}\left(\frac{3}{4}\right)^{6} + {}^{8}C_{3}\left(\frac{1}{4}\right)^{3}\left(\frac{3}{4}\right)^{5} =$$

$= 0.10011 + 0.26698 + 0.31146 + 0.20764$

$= \underline{\underline{0.88619}}$

Example 10

A large consignment of manufactured articles is accepted if either of the following conditions is satisfied:

i) A random sample of 10 articles contains no defective articles

ii) A random sample of 10 contains one defective article and a second random sample of 10 is then drawn which contains no defective articles.

Otherwise the consignment is rejected. If 5% of the articles in a given consignment are defective, what is the probability that the consignment is accepted?

Solution

$$P(defective) = \frac{5}{100} = 0.05 = p$$

$$P(non\text{-}defective) = 0.95 = q$$

$$n = 10$$

Condition (i)

10

$P(x=0) = {}^{10}C_0(0.5)^0(0.95)^{10} = \underline{\underline{0.5987}}$

Condition (ii)

$P(x=1) \cdot P(x=0)$

$= {}^{10}C_1(0.05)^1(0.95)^9 \times 0.5987$

$= 0.3151 \text{ x } 0.5987 = \underline{\underline{0.1887}}$

Therefore the probability of accepting the consignment = (i) or (ii)

$= 0.5987 + 0.1887$

$= 0.7874$

Exercises

1. A fair coin is tossed six times. Find the probability of obtaining at least four heads.

2. 30% of pupils in a school travel to school by bus. From a sample of 8 pupils chosen at random, find the probability that

(a) only three travel by bus

(b) less than half travel by bus

3. A fair die is thrown 7 times. Find the probability of throwing more than 4 sizes.

4. A fair coin is tossed 12 times. Find the probability of obtaining 7 tails.

5. The probability of an arrow hitting a target is 0.9. Find the probability of at least 3 arrows hitting the target if 5 arrows are shot.

6. If there are 10 traffic lights along a certain road, find the probability of getting 2 red lights if the probability of a red light is $\frac{2}{5}$.

CHAPTER FOUR

POISSON DISTRIBUTION

INTRODUCTION

Sometimes, one may be interested in occurrences in a specified time period, length, area or volume. For example, at certain times of the year there are virtually no accidents whilst more traffic accidents are recorded on occasions like Easter, Christmas, National holidays, Ramadan festivals. One may therefore be interested in finding the probability of a number of traffic accidents, which would occur, in a festive period or a fraction of a festive period. It is also known that at certain times of the week one finds a large number of airplanes at Kotoka International Airport whilst at other times, no planes are found at the airport.

It is known that at certain times, e.g. some few days after pay day, one finds a large number of customers in a queue at the cashier‟s counter in banks, whilst at other times, the banks are virtually empty, and therefore no customer at the cashier‟s counter.

Other examples of such variables are; the number of mistakes on a paper of a book, number of radio-active elements detected by a Geigercounter; number of ships arriving at an harbour in a particular time period; number of flaws in a textile (cloth); number of fire outbreaks in a given period of time; the number of breakdowns of machines per year; number of telephone calls on Monday between 8am and 9am.

These events have certain characteristics.

i. The events occur at random in continuous space or time.

ii. The events occur singly, and the probability of two or more events occurring at the same time is zero.

iii. The events occur uniformly. Thus the expected number of events in a given interval is proportional to the size of the interval.

iv. The events are independent.

v. The variable is the NUMBER of events that occur in an interval of a given size.

DEFINITION:

The number x of successes in a Poisson experiment is called a Poisson random variable. The probability distribution of the Poisson variable, x is

$$P(x,\lambda) = \frac{e^{-\lambda}\lambda^{x}}{x!}$$

where $x = 0,1,2,3\ldots$ and λ is the average number of successes in the given time interval or length, or space or volume. The Poisson distribution is completely defined by its mean, $\lambda = np$. It is an approximation to the Binomial when the number of trials is large ($n > 30$) but $np < 5$.

The mean, λ, depends on the size of the stated unit. It changes proportionally whenever the stated unit is changed.

The Poisson distribution is given by

$$P(X = x) = \frac{e^{-\lambda}\lambda^{x}}{x!}, \qquad x = 0, 1, 2, \ldots$$

$$P(X \leq \infty) = \sum_{x=0}^{\infty} \frac{e^{-\lambda}\lambda^{x}}{x!}$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{\lambda^{x}}{x!}$$

$$= e^{-\lambda}\left(1 + \lambda + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \dots\right)$$

$$= e^{-\lambda}e^{\lambda}$$

= 1

Therefore $P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!}$ is a probability distribution.

The expected value of the Poisson distribution = λ.

The variance of the Poisson distribution is also λ. This is the only distribution whose mean and variance are the same.

Example 1

Given that X is a Poisson variable with mean, $\lambda = 2$.

Calculate the following probabilities:

(i) $P(X = 0)$

(ii) $P(X \,\square\, 2)$

(iii) $P(X \,\square\, 1)$

(iv) $P(X \,\square\, 2)$

Solution

(i) $$P(X = 0) = \frac{e^{-2}2^0}{0!} = e^{-2} = 0.1353$$

(ii) $$P(X = 2) = \frac{e^{-2}2^2}{2!} = 2e^{-2} = 0.2706$$

(iii) $$P(X > 1) = 1 - P(x \le 1)$$

$$= 1 - \left[P(X = 0) + P(X = 1)\right]$$

$$= 1 - \left[0.1353 + \frac{e^{-2}2^1}{1!}\right]$$

$$= 1 - \left[0.1353 + 0.2706\right]$$

$$= 0.5941$$

(iv) $$P(X > 2) = 1 - P(x \le 2)$$

$$= 1 - \left[P(X = 0) + P(X = 1) + P(X = 2)\right]$$

$$= 1 - \left[0.1353 + 0.2706 + 0.2706\right]$$

$$= 0.3235$$

Example 2

The probability that an insurance company must pay a particular medical claim for a policy is 0.001. If the company has 1000 of such policyholders, what is the probability that the insurance company will have to pay at least 2 claims?

Solution

This is a typical Binomial variable. However with $n = 1000$ $(n > 30)$ and $np = 1$ $(np < 5)$ we use the Poisson approximation.

Thus $\lambda = np$

$= 1000 \times (0.001)$

$= 1$

$$P(X \geq 2) = 1 - P(X \leq 1)$$

$$= 1 - [P(X = 0) + P(X = 1)]$$

$$= 1 - \left[\frac{e^{-1}1^0}{0!} + \frac{e^{-1}1^1}{1!}\right]$$

$$= 1 - 2e^{-1}$$

$$= 0.2642$$

Example 3

2% of bulbs produced by a machine are defective. In a random sample of 100 bulbs produced by the machine, find the probability that it will include:

(i) Exactly 1

(ii) Exactly 2

(iii) More than 3 defective bulbs.

Solution

$$n = 100, \quad \lambda = np$$

$$= 100 \times \frac{2}{100}$$

$$= 2$$

(i) $$P(X = 1) = \frac{e^{-2}\, 2^{1}}{1!} = 0.2707$$

(ii) $$P(X = 2) = \frac{e^{-2}\, 2^{2}}{2!} = 0.2707$$

$$P(\text{more than } 3 \text{ defective bulbs}) = 1 - \left[P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3)\right]$$

$$P(X = 0) = \frac{e^{-2}\, 2^{0}}{0!} = e^{-2} = 0.1353$$

$$P(X = 3) = \frac{e^{-2}\, 2^{3}}{3!} = 0.1804$$

Hence the required probability $= 0.1353 + 0.2707 + 0.2707 + 0.1804$

 0.8571

Example 5

Suppose there is an average of 10 fire outbreaks in 20 days of a certain month. What is the expected number of fire outbreaks in 10 days of that month?

Solution

In 20 days, mean number = 10

Therefore, in 10 days, mean number = $\frac{10}{20} \times 10 = 5$

Thus having the stated number also halves the expected number. Similarly, doubling the expected number unit also doubles the expected number of occurrences for the new stated unit.

Example 4

The expected number of telephone calls is 6 per minute.

What is the probability of getting (a)

4 calls in the next two minutes?

(b) 2 calls in the next thirty seconds?

Solution

(a) New mean, $\lambda = \frac{2}{1} \times 6 = 12$ *calls*/2mins

$$P(X = 4) = \frac{e^{-\lambda}\lambda^x}{x!}$$

$$= \frac{e^{-12}12^4}{4!}$$

$$= 0.0053$$

(b) The new mean, $\lambda = \frac{30}{60} \times 6 = 3$ *calls* /30mins

Therefore $P(X = 2) = \frac{e^{-3}3^2}{2!} = 0.2240$

Example 5

Suppose there is an average of 10 fire outbreaks in 20 days of a certain month. What is the expected number of fire outbreaks in 10 days of that month?

Solution

In 20 days, mean number = 10

Therefore, in 10 days, mean number = $\frac{10}{20} \times 10 = 5$

Thus having the stated number also halves the expected number. Similarly, doubling the expected number unit also doubles the expected number of occurrences for the new stated unit.

Example 6

A typist averages two errors per page of a report. Assuming that the number of errors is a Poisson variable, calculate the probability that

(i) Exactly two errors will be found on a given page of the report.

(ii) At most three errors will be found on any two pages of the report.

Solution:

(i) The stated units are the same

Thus λ = 2/page

$$P(x = 2) = \frac{e^{-2}2^2}{2!}$$

$$= 2e^{-2}$$

$$= 0.2706$$

(ii) The stated units have changed.

New mean, $\lambda = \frac{2}{1} \times 2 = 4$ errors/2 pages

$P(x \leq 3) = P(x = 0) + P(x = 1) + P(x = 2) + P(x = 3)$

$$= \frac{e^{-4}4^0}{0!} + \frac{e^{-4}4^1}{1!} + \frac{e^{-4}4^2}{2!} + \frac{e^{-4}4^3}{3!}$$

$= e^{-4} (1 + 4 + 8 + 10.667)$

$= 23.667e^{-4}$

$= (23.667)\ (0.0183)$

$= 0.4331$

<u>Exercises:</u>

1. (a) The fire department in a city can put out a fire in 1 hour, and the average is 2.4.

   i. What is the probability that no alarms are received for 1 hour? ii.

   What is the probability that no alarms are received in 2 hours? iii.

   What is the most probable number of alarms in an hour?

2. If a keyboard operator averages two errors per page of newsprint, and if these errors follow Poisson distribution, what is the probability that

   i. Exactly four errors will be found on a given page?

   ii. At least two errors will be found?

3. The number of misprints in a book is Poisson distributed with an average of 5 in 10 pages. What is the probability of getting at most 1 error on a page of a book? If 5 pages of the book are selected at random, what is the probability of getting at least 2 pages with at most1 error on a page?

4. A car hire company finds that over a period, the expected demand for cars is 2 per working day. The cars are hired for one-day return journeys only.

Assuming that the demand for cars is a Poisson variable,

i. Calculate the probability that there is no demand for cars on any working day.

ii. If the company has two cars, what is the probability that the company cannot cope with demand?

iii. The company makes a profit of ¢15,000.00 per day when a car is out for hire and loses ¢10,000.00 each day a car is not used. Would it be profitable for the company to buy another car if the average demand for cars remained at 2 per day?

5. The number of misprints in a book is Poisson distributed with an average of 5 in 10 pages. What is the probability of getting at most 1 error on a page of a book?

If 10 pages of the book are selected at random, what is the probability of getting at least 2 pages with at most 1 error?

CHAPTER FIVE

NORMAL DISTRIBUTION

INTRODUCTION

This is one of the most widely used continuous probability distributions. Like all continuous distributions, probabilities of occurrence of continuous variables are computed as a ratio of length of a section to total length, volume of a section to total volume or area to total area. For a continuous variable, the probability at any point is

ZERO i.e. $P(X = c) = 0$

All normal distributions have the same "bell –shape". They are symmetrical about the mean. The scores range from negative infinity through zero to plus infinity ($-\infty < X < +\infty$)

There are an infinite number of normal distributions. Each normal distribution is completely defined if the **MEAN** and **STANDARD DEVIATION** are specified.

It has the probability function

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

One special normal distribution is of special interest. This is the **UNIT NORMAL**

UNIT NORMAL DISTRIBUTION

The NORMAL DISTRIBUTION has a mean, μ=0 and a standard deviation, σ = 1. The scores range between all positive and negative Numbers i.e. $(-\infty < Z < \infty)$.

The scores are usually referred to as STANDARD SCORES. The mean, median and mode coincide at 0. i.e. It is symmetric about □ = 0. The total area under the curve =1. It has the probability function

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

$$P(a \leq Z \leq b) = \frac{\text{area under the curve for that interval}}{\text{Total area under the curve}}$$

= Area under the curve (since the total area under the curve = 1)

The total probability of a score, a, from – ∞ and including that Z value

(a) = $P(-\infty < Z \leq a)$. has been computed and presented in the form of a table often called the Normal Distribution Table.

Example 1

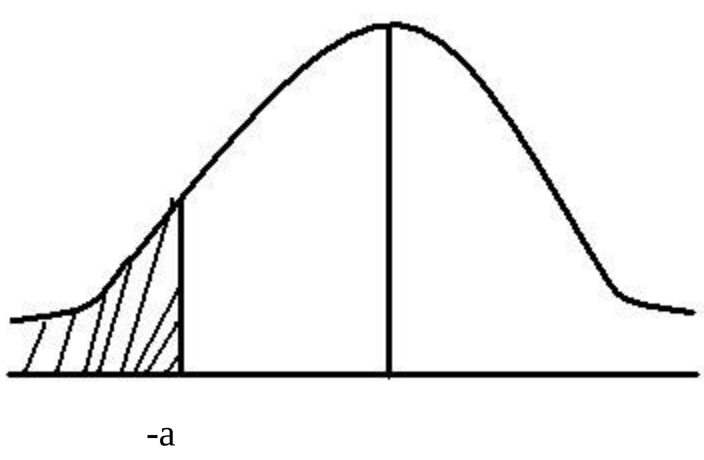
$P(Z < 2)$ = 0.9772; $P(Z < 1.5)$=0.9332

You must have realized that although the range of standard scores is from negative infinity to positive infinity, the values of $\Phi(a) = P(-\infty < Z < a)$ have been provided for only positive scores. There is an easy method of evaluating $P(Z < -a)$ say.

The unit normal distribution is symmetrical about the mean $\mu = 0$. Thus area in the interval $(-a, 0)$ is the same as the area in the interval $(0, a)$. Thus P (Z < -a) = P (Z>a) =1-P (Z<a) (because the total area under the curve = 1)

This gives a simple relationship between P (Z< a) and P (Z < -a) = 1- P (Z <a).

i.e. $\Phi(-a) = 1 - \Phi(a)$



NOTE

P (Z < - 2) = 1 - P (Z < 2) i.e. $\Phi(-2) = 1 - \Phi(2)$

$P(Z < -1.5) = 1 - P(Z < 1.5)$ i.e. $\Phi(-1.5) = 1 - \Phi(1.5)$

Example 2

Find i) $P(Z <- 0.25)$; ii) $P(Z- < 0.05)$; iii) $P(Z < -0.5)$

Solution

i) $P(Z <- 0.25) = 1 - P(Z< 0.25)$

$= 1 - \Phi(0.25)$

$= 1 - 0.5987$

$= 0.4013$ ii) P

$(Z-<0.05) = 1 - P(Z< 0.05)$

$= 1 - \Phi(0.05)$

$= 1 - 0.5199$

$= 0.4801$

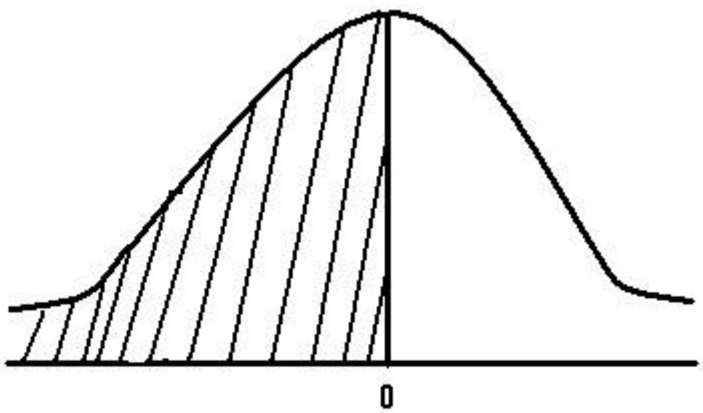
iii) $P(Z<-0.5) = 1 - P(Z< 0.5)$

$= 1 - \Phi(0.5)$

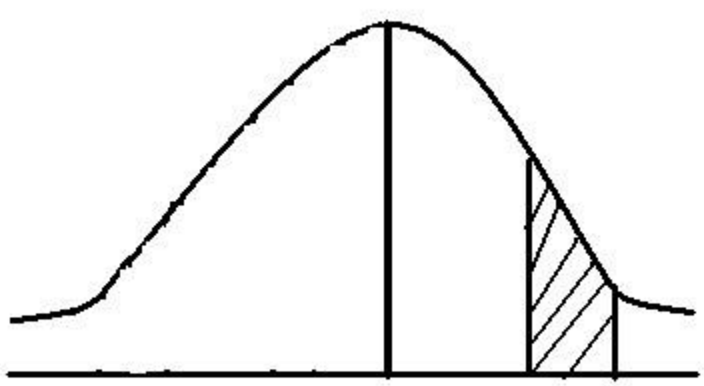
$= 1 - 0.6915$

$= 0.3085$

Note that the probability of a negative value on the UNIT NORMAL SCALE is less than 0.5



Let us now compute P (a< Z< b)



0 a b

Thus $P(a < Z < b) = \Phi(b) - \Phi(a)$

Example 3

$P(1 < Z < 2) = \Phi(2) - \Phi(1)$

Solution

$\Phi(2) - \Phi(1) = 0.9772 - 0.8451$

$= 0.1321$

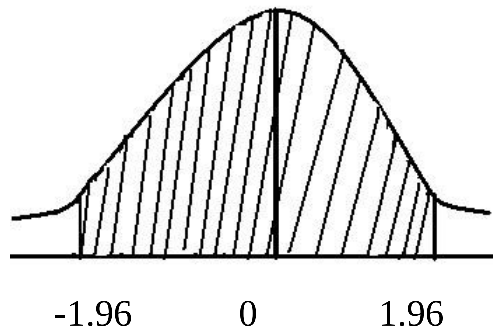
Example 4

$P(-1.96 < Z < 1.96.) = \Phi(1.96) - \Phi(-1.96)$

$= \Phi(1.96) - [1-\Phi(1.96)]$

$= \Phi(1.96) - 1 + \Phi(1.96)$

$= \Phi(1.96) - [1-\Phi(1.96)]$



$= 2\Phi(1.96) - 1$

$= \quad 0.950$

The unit normal table is also used to find Z scores of given probabilities.

Example 5

Find c such that:

i) $P(Z < c) = 0.95$ ii)

$P(Z<c) = 0.1587.$

iii) $P(Z < c) = 0.05$

Solution

(i) We search for the probability of 0.95 and read off the corresponding Z value as c.

Thus $c = \Phi^{-1}(0.95) = 1.64$ or 1.65.

(ii) We note in this example that the probability 0.1587 is less than 0.5.

Probabilities less than 0.5 are not listed in the table.

These are obtained as $c = -\ \square^{-1}\ (1 - 0.1587)$

$= -\ \square^{-1}(0.8413) = -\ 1.00$ iii) $c = -\ \square^{-1}(1 -$

$0.05) = -\ \square^{-1}\ (0.95) = -1.645$

Example 6

1. Find the following probabilities with the help of the unit Normal table.

(a) $P(-1.96<Z<2.33)$

(b) $P(Z>-0.5)$

(c) $P(-1.96<Z<1.96)$

2. Find a, such that

a) $P(0<Z<a) = 0.3413$

b) $P(Z>-a) = 0.6554$

c) $P(|Z|<a) = 0.226$

3. Find a, if

a) $P\ (|Z|<2.57) = a$

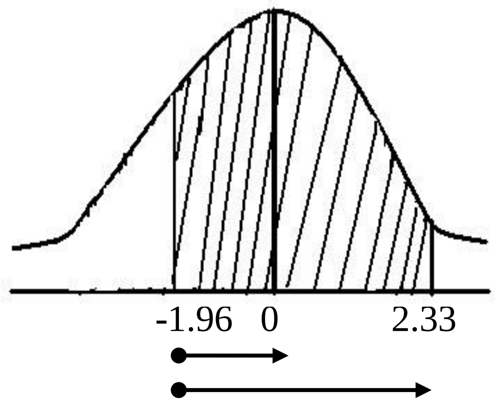
b) $P(|Z|>2.57) = a$

Solutions

FINANCIAL OTABIL

1.

(a) $P(-1.96<Z<2.33) \quad =\Phi(2.33)-\Phi(-1.96)$

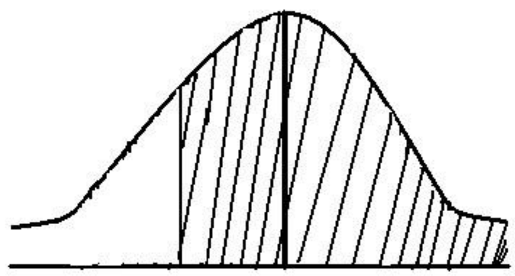


$= \Phi(2.33)-[1-\Phi(1.96)]$

$=0.9901-[1-0975]$

$=0.9651$

(b) $P(Z>-0.5) =1-P(Z<-0.5)$



$=1-\Phi(-0.5)$

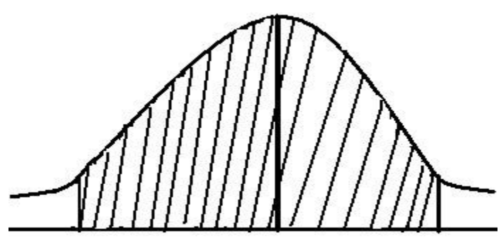
$=1-[1-\Phi(0.5)]$

$= \Phi(0.5)$ -0.5

0

$=0.6915$

(c) $P(-1.96<Z<1.96)= \Phi(1.96)- \Phi(-1.96)$



$= \Phi(1.96)-[1-\Phi(1.96)]$

$= (0.9750)-[1-0.9750]$ -1.96 0 1.96

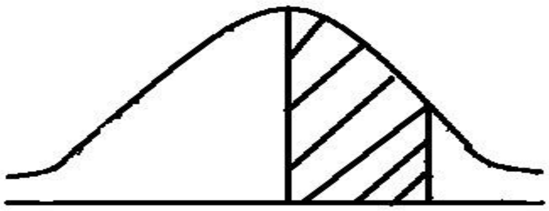
$=0.95$

2. (a) $P\ (0<Z<a)=0.3413$

$\Phi(a)- \Phi(0) =0.3413$

$\Phi(a)- \Phi(0) =0.3413$

$\Phi(a)-0.5 = 0.3413$



0 a

$\Phi(a) = 0.3415+0.5$

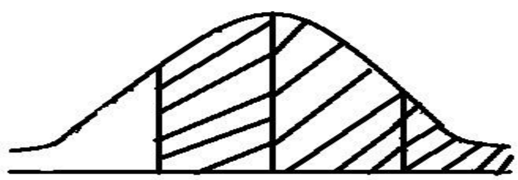
$= 0.8415$

Therefore $a=\Phi^{-1}(0.8415) = 1$

2 (b) $P(Z>-a) = 0.6554$

$= 1-P(Z<-a)$

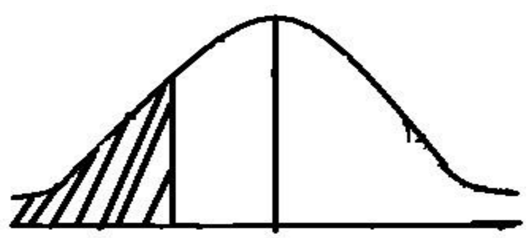
$= 1-[1-\Phi(a)]$



$\Phi(a)=0.6554$

$\therefore$ $a = \Phi^{-1}(0.6554) = 0.4$ 2 (c) $P(Z<a) = 0.226$ $a = -\Phi^{-1}(1-0.226)$

$a = -\Phi^{-1}(0.7740)$



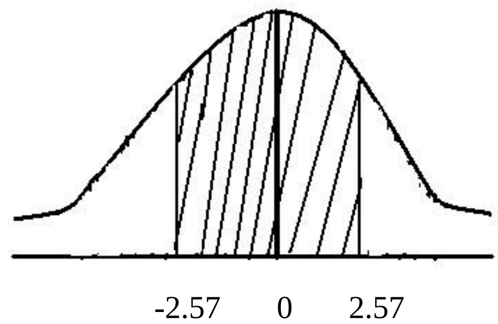
a = -<u>0.75</u>

3 (a) P (|Z|<2.57) =a a = P (-2.57<Z<2.57)

a = □(2.57) - □(-2.57)

a = 0.9949-[1-0.9949]

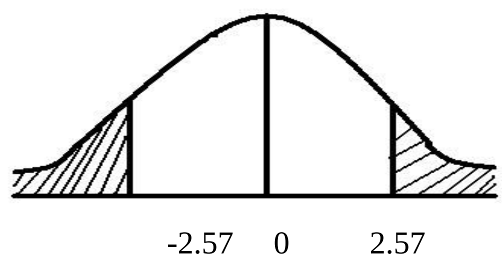
a = 0.990



3 (b) P (|Z|>2.57) =P (Z>2.57) or P (Z<-2.57)

P (|Z|>2.57) = P (Z>2.57) + P (Z<-2.57)

= [1-P (Z<2.57)] + [1-P (Z<2.57)]



= 2 [1-P (Z<2.57)]

= 2 [1-0.9949]

= 0.0102

STANDARD SCORES

Now that we know of the Unit Normal distribution and how to use the unit normal distribution table, let us extend it to other normal distributions. There is an infinite number of normal distribution, each one completely defined by its mean, □, and standard deviation, □. Fortunately, all normal distributions have a simple algebraic relationship with the unit normal distribution. Suppose we have a normal distribution with scores, X_i, mean μ and standard deviation, σ.

Each score, X_i, can be transformed into a standard score, Z_i, by use of the linear relationship

$$\boxed{Z_i = \frac{X_i - \mu}{\sigma}}$$

The variable, Z_i, will have a mean, $\mu = 0$ and a standard deviation, $\sigma = 1$.

Once the value has been transformed to a standard score, its distribution becomes that of the unit normal distribution.

Example 1

The marks scored by some students in an examination are normally distributed with a mean, $\mu = 6$ and a standard deviation, $\sigma = 2$. What percentage of the students scored between 5 and 10?

Solution

Here the mean, $\mu = 6$ and the standard deviation, $\sigma = 2$.

We shall therefore have to transform the values 5 and 10 to standard score before we use the Unit Normal table.

$$P(5 < x < 10) = P\left(\frac{5 - 6}{2} < \frac{X - \mu}{\sigma} < \frac{10 - 6}{2}\right)$$

—— □ ——— 5□26 *Z* 10□6 □□

□□□ □ □ 2 □

□□0.5 □ □*Z* 2□

□□□2□□□□□0.5□

□ 0.9772□ □□□1

□□0.5□□□

□ 0.9772□ □□10.6915□

□ 0.6687

Therefore 66.87 % of the data fall between 5 and 10.

Example 2

A random variable X is normally distributed with mean □ = 5 and □ = 2

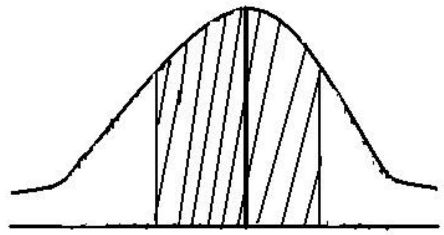
i.e. #

X□N (5, 2)

Find i) P (4<X<6) ii) P (X>8)

Solution

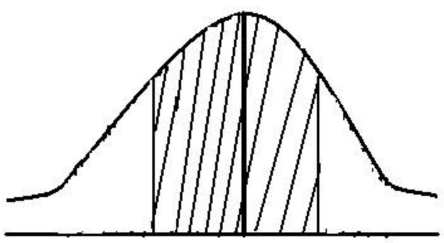
i) P (4<X<6)



$$P(4 < X < 6) = P\left(\frac{4-\mu}{\sigma} < \frac{X-\mu}{\sigma} < \frac{6-\mu}{\sigma}\right)$$

$$P\left(\frac{4-5}{2} < Z < \frac{4-5}{2}\right) = \Phi\ (-0.5<Z<0.5)$$

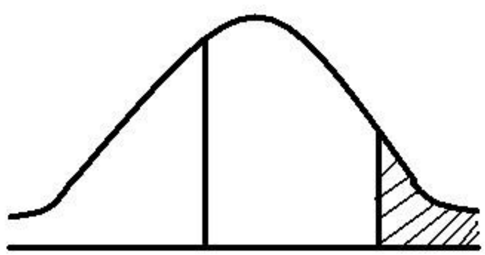
$= \Phi(0.5) - \Phi(-0.5) = \underline{0.3830}$



ii) P (X>8)

=1 - P(X<8)

$= 1 - \frac{4}{16}$



$$= 1 - p\left(Z < \frac{8-5}{2}\right)$$

$= 1 - \Phi(1.5)$

= 0.0668

Example 3

1

1. If X□N (10, 2^2), find

i) P(7 <X< 13) ii)

P(X > 13)

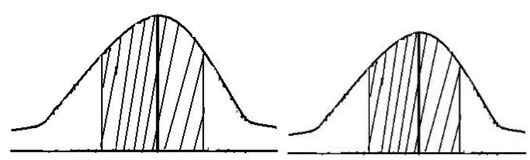
2. If X□ N(60, 6^2), Find

i) P(X=31)

ii) P(|X-60|>9)

3. Suppose that the height of some Students are normally distributed with mean □=65 inches and variance 9 inch squared. What percentage of them has heights between 59 inches and 71 inches?

Solution



7 10

i) P (7<X<13)

13 p□□ _____□^{2} □ □Z______□^{2} □□□ =□ (1.5) - 10

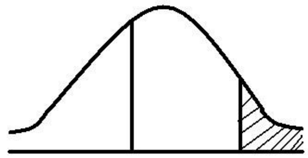
□(-1.5)

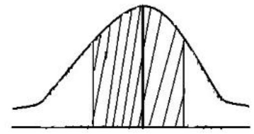
= □

= 0.8664 ii) P(X>13) = 1-P(X<13)

= 1 -□ (1.5)

=<u>0.668</u>





1

2. i) P (Z=31) = 0 for a continuous variable P (X=C)=0

ii) P (X<50) = *p Z*□□□ □______54 60□6 □□□

=□ (-1)

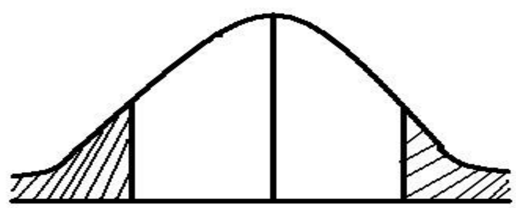
=0.1587 iii) |x-60|>9 =

(X-60)>9

= X > 69 or

-(X-60)>9 = -X + 60 > 9

=51 > X



Thus | X-60|>9

= X<51 or X > 69

Therefore P (|X-60|> 9)

= P (X > 69) +P (X< 51) =

{1- P (X<69)} + P (X< 51)

= 1□□□□□______69 60□6 □□□□□□□□______51 60□6 □□□

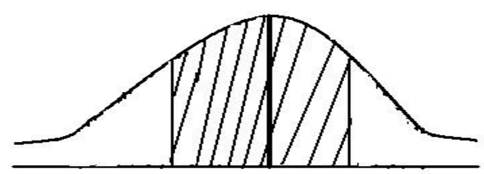
=1 - □ (1.5) + □ (- 1.5)

=1- □ (1.5) + [1-□(1.5)]

=0.0668 + 0.0668

= 0.1336

3.) X□ N (65,9)



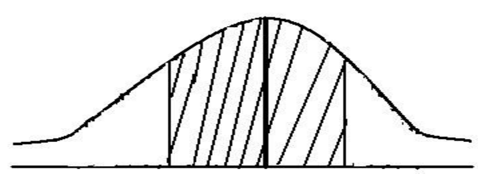
59 65 75

□71 65□□ □59 65□□

□ 3 □□- □□□ 3 □□

P (59<X<71) = □□

= □ (2) - □(-2)



-2 0 2

= <u>0.9544</u>

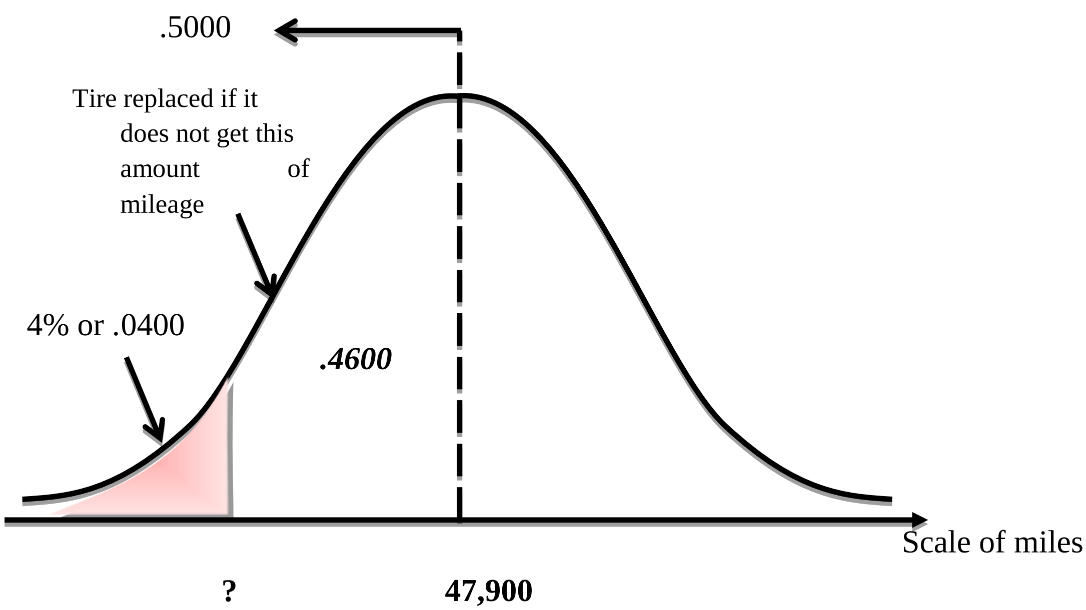
Therefore 95% of the students with heights between 59 inches and 71 inches.

Example 4

Suppose a tire manufacturer wants to set a minimum mileage guarantee on its new MX 100 tire. Tests reveal the mean mileage is 47,900 with a standard deviation of 2,050 miles and a normal distribution. The manufacturer wants to set the minimum guarantee mileage so that no more than 4% of the tires will have to be replaced. What minimum guarantee mileage should the manufacturer announce?

Solution

The facets of this problem are shown in the following diagram, where *x* represents the minimum guarantee mileage.



Inserting these values in the formula $\left(Z = \frac{X-\mu}{\sigma}\right)$ for *z*:

$$Z = \frac{X-\mu}{\sigma} = \frac{X - 47{,}900}{2{,}050}.$$

There are two unknowns, *z* and X. To find *z*, notice the area under the normal curve to the left of μ is .5000. The area between μ and X is .4600, found by (.5000 – .0400). (Now refer to the tables of last pages). Search the body of the table for the area closers to .4600, namely .4599.

Move to the margins from this value and read the *z* value of 1.75. Because the values is to the left of the mean, it is actually –1.75. These steps are summarized in the table below:

Z	.03	.04	.05	.06
…	…	…	…	…

1.5	.4370	.4382	.4394	.4406
1.6	.4484	.4382	.4505	.4515
1.7	.4582	.4591	.4599	.4608
1.8	.4664	.4671	.4678	.4686

Knowing that the distance between μ and X is -1.75σ, we can now solve for X (the minimum guaranteed mileage):

$$z = \frac{X - 47{,}900}{2{,}050}$$

$$-1.75 = \frac{X - 47{,}900}{2{,}050}$$

$$-1.75(2{,}050) = X - 47{,}900$$

$$X = 47{,}900 - 1.75(2{,}050) = 44{,}312$$

So the manufacturer can advertise that it will replace for free any tire that wears out before it reaches 44,312 miles, and the company will know that only 4 percent of the tires will be replaced under this plan.

Example 5

Suppose a study of the inmates of a correctional institution is concerned with the social responsibility of the inmates in prison and their prospects for rehabilitation upon being released. Each inmate is given a test regarding social responsibility. The scores are normally distributed, with a mean of 100 and a standard deviation of 20. Prison psychologists rated each of the inmates with respect to the prospect for rehabilitation. These ratings are also normally distributed, with a mean of 500 and a standard deviation of 100. Tora Carney scored 146 on the social responsibility test, and her rating with respect to rehabilitation is 335. How does Tora compare to the group with respect to social responsibility and the prospect for rehabilitation?

Solution

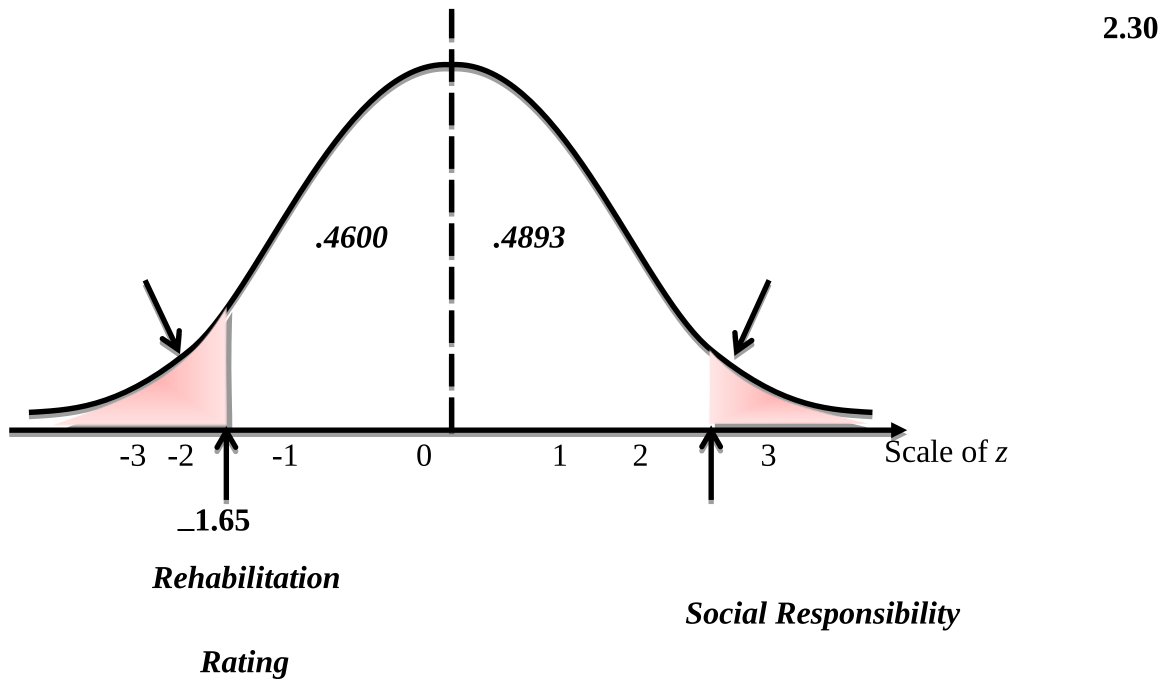
Converting her social responsibility test score of 146 to a *z* value using formula

$$z = \frac{X - \mu}{\sigma} = \frac{146 - 100}{20} = \frac{46}{20} = 2.30$$

Converting her rehabilitation rating of 335 to a *z* value:

$$z = \frac{X - \mu}{\sigma} = \frac{335 - 500}{100} = \frac{-156}{100}$$

The Standardized test score and the standardized rating are shown below.



With respect to social responsibility, therefore, Tora Carney is in the highest 1% of the group. However, compared with the other inmates, she is among the lowest 5% with regard to the prospects for rehabilitation.

EXERCISES

1. If IQ scores of some people are normally distributed with a mean of 100 and a standard deviation of 15, what proportion of the people have IQ‟s: a) above 110; b) above 125; c) below 80; d) above 75; e) between 100 and 115; f) between 75 and 125; g) between 135 and 145; h) between 60 and 90?

2. In relation to the distribution of IQ‟s in Question (1), assume that it is the practice to provide special education for the lowest 5% of the population, and provide university education for the top 7%. Find the z-scores (i.e. standard normal scores)corresponding to these percentages and hence state

what you would expect would be the IQ cut-off points for those requiring special education, and those entering university.

3. The mean diameter of a sample of washers produced by a machine is 5.02mm and the standard deviation is 0.05mm. The tolerance limits of the diameter is 4.96mm to 5.08mm otherwise it is rejected. It 1000 washers were produced how many would be expected to be rejected?

4. The mass of eggs laid by some hens are normally distributed with mean 60 grams and standard deviation 15 grams. Egg"s mass less than 45 grams are classified as small. The remainders are divided into standard and large, and it is desired that these should occur with equal frequency. Suggest the mass at which the division should be made (correct to the nearest gram).

5. The weights of bars of soap made in a factory are normally distributed. Last week $6^2/_3$% of bars weighed less than 90.50 grams and 4% weighed more than 100.25 grams.

   Find the mean and variance of the distribution of weights, and the percentage of bars produced, which would be expected to weigh less than 88 grams.

   If the variance of the weight distribution was reduced by one-third, what percentage of the next week"s production would you expect to weigh less than 88 grams, assuming the mean is not changed?

6. The heights of students in Ghana are normally distributed with a mean of 62 inches and a standard deviation of 8 inches. How long should

mattresses produced from a factory be in order to accommodate 95 per cent of them?

7. The weights of some students are normally distributed with mean 80 kg and a standard deviation 20 kg. The students are classified into groups A, B and C by weight. 20% of the students belong to group C and groups A and B have equal proportions of students. Obtain the weights, which divide the students if those in group A weigh, less than those in group B and those in group C are the heaviest students.

CHAPTER SIX

FORECASTING

INTRODUCTION

Every day, managers make decisions without knowing what will happen in the future. Inventory is ordered though no one knows what sales will be, new equipment is purchased though no one knows the demand for products, and investments are made though no one knows what profits will be. Managers are always trying to reduce this uncertainty and to make better estimates of what will happen in the future. Accomplishing this is the main purpose of forecasting.

There are many ways to forecast the future. In numerous firms (especially smaller ones), the entire process is subjective, involving seat-of-the pants methods, intuition, and years of experience. there are also many quantitative forecasting models, such as moving averages, exponential smoothing, trend projections, and least squares regression analysis,

Regardless of the method that is used to make the forecast, the same eight overall procedures that follow are used.

QUALITATIVE MODELS

Whereas time-series and causal models rely on quantitative data, *qualitative models* attempt to incorporate judgmental or subjective factors into the forecasting model Opinions by experts, individual experiences and judgments, and other subjective factors may be considered.

Qualitative models are especially useful when subjective factors are expected to be very important or when accurate quantitative data are difficult to obtain.

Here is a brief overview of four different qualitative forecasting techniques:

Delphi method: This iterative group process allows experts, who may be located in different places, to make forecasts. There are three different types of participants in the Delphi process: decisions makers, staff personnel, and respondents. The *decision making group* usually consists of 5 to 10 experts who will be making the actual forecast. The staff personnel assist the decision makers by preparing, distributing, collecting, and summarizing a series of questionnaires and survey results. The respondents are a group of people whose judgments are valued and are being sought. This group provides inputs to the decision makers before the forecasts are made.

Jury of executive opinion: This method takes the opinions of a small group of high level managers, often in combination with statistical models, and results in a group estimate of demand.

Sales force composite: In this approach, each salesperson estimates what sales will be in his or her region; these forecasts are reviewed to ensure that they are realistic and are then combined at the district and national levels to reach an overall forecast.

Consumer market survey: This method solicits input from customers or potential customers regarding their future purchasing plans. It can help not only in preparing a forecast but also in improving product design and planning for new products.

MEASURES OF FORECAST ACCURACY

We discuss several different forecasting models in this chapter. To see how well one model works, or to compare with the actual or observed values. The forecast error (or deviation) is defined as follows:

Forecast error = actual value – forecast value

One measure of accuracy is the *mean absolute deviation* (MAD). This is computed by taking the sum of the absolute values of the individual forecast errors and dividing by the numbers of errors (n);

$$MAD = \frac{\sum |\text{forecast error}|}{n}$$

Consider the Wacker Distributors sales of CD players seen in Table 5.1. Suppose that in the past, Wacker had forecast sales for each year to be the sales that were actually achieved in the previous year. This is sometimes called a *naïve* model. Table 5.2 gives these forecasts as well as the absolute value of the errors in forecasting for the next time period (year 11), the forecast would be 190. Notice that there is no error computed for year 1 since there was no forecast for this year, and there is no error for year 11 since the actual value of this is not yet known.

Thus, the number of errors *(n)* is 9.

From this, we see the following:

$$MAD = \frac{\sum |\text{forecast error}|}{n} = \frac{160}{9} = 17.8$$

This means that on the average, each forecast missed the actual value by 17.8 units.

YEAR **ACTUAL** **FORECAST** **ABSOLUTE VALUE SALES OF CD SALES OF ERRORS PLAYERS (DEVIATION)**

[ACTUAL – FORECAST]

1 110 - -

2 100 110 |100-110| = 10 **3** 120 100 |120-

100| = 20 **4** 140 120 |140-120| = 20 **5** 170

140 |170-140| = 30 **6** 150 170 |150-170| =

20 **7** 160 150 |160-150| = 10 **8** 190 160

|190-160| = 30 **9** 200 190 |200-190| = 10

10 190 200 |190-200| = 10

11 - 190 -

Sum of |errors| = 160

MAD = 160/9 = 17.8

There are other measures of the accuracy of historical errors in forecasting that are sometimes used besides the MAD. One of the most common is the *mean squared error* (MSE) which is the average of the squared errors:

$$MSE = \frac{\sum (errors)^2}{n}$$

Besides the MAD and MSE, the mean *absolute percent error* (MAPE) is sometimes used. The MAPE is the average of the absolute values of the errors expressed as percentages of the actual values. This is computed as follows:

$$MAPE = \frac{\sum \left| \frac{error}{actual} \right| 100\%}{n}$$

There is another common term associated with error in forecasting. Bias is the average error and tells whether the forecast tends to be too high or too low and by how much. Thus, bias may be negative or positive. It is not a good measure of the actual size of the errors because the negative errors can cancel out the positive errors.

Moving Averages

Moving averages are useful if we can assume that market demands will stay fairly steady over time. For example, a four months and dividing by 4. With each passing month, the most recent month‟s data are added to the sum of the previous three months‟ data, and the earliest month is dropped. This tends to smooth out short-term irregularities in the data series. An *n*-period moving average forecast, which serves as an estimate of the next period‟s demand, is expressed as follows:

$$\textit{moving average forecast} = \frac{\text{sum of demands in previous n periods}}{n}$$

Mathematically, this is written as

$$F_t = \frac{Y_{t-1} + Y_{t-2} + \ldots + Y_{t-n}}{n}$$

where

$F_t =$ forecast for time period t
$Y_t =$ actual value in time period t
$n =$ number of periods to average

A four month moving average has $n = 4$; a five-month moving average has $n = 5$.

Wallace Garden Supply Example

Storage shed sales at Wallace Garden Supply are shown in the middle column of Table 7.3. A 3-month moving average is indicated on the right. The forecast for the next January, using this technique, is 16. Were we simply asked to find a forecast for next January; we would only have to make this one calculation. The other forecasts are necessary only if we wish to compute the MAD or another measure of accuracy.

When there might be a trend or pattern emerging, weights can be used to place more emphasis on recent values. This makes the technique more responsive to changes because latter periods may be more heavily weighted. Deciding which weights to use requires some formula to determine them. However, several different sets of weights to use require some experience and a bit of luck. Choice of weights to use requires some formula to determine them. However, several different sets of weights may be tried, and the average is that if the last month or period is weighted too heavily, the forecast might predict a large unusual change in the demand or sales pattern too quickly, when in fact the change is due to random fluctuation.

A weighted moving average may be expressed as

$$\textit{weighted moving average} = \frac{\sum (\text{weight for period } n)(\text{demand in period } n)}{\sum \text{weights}}$$

Mathematically this is

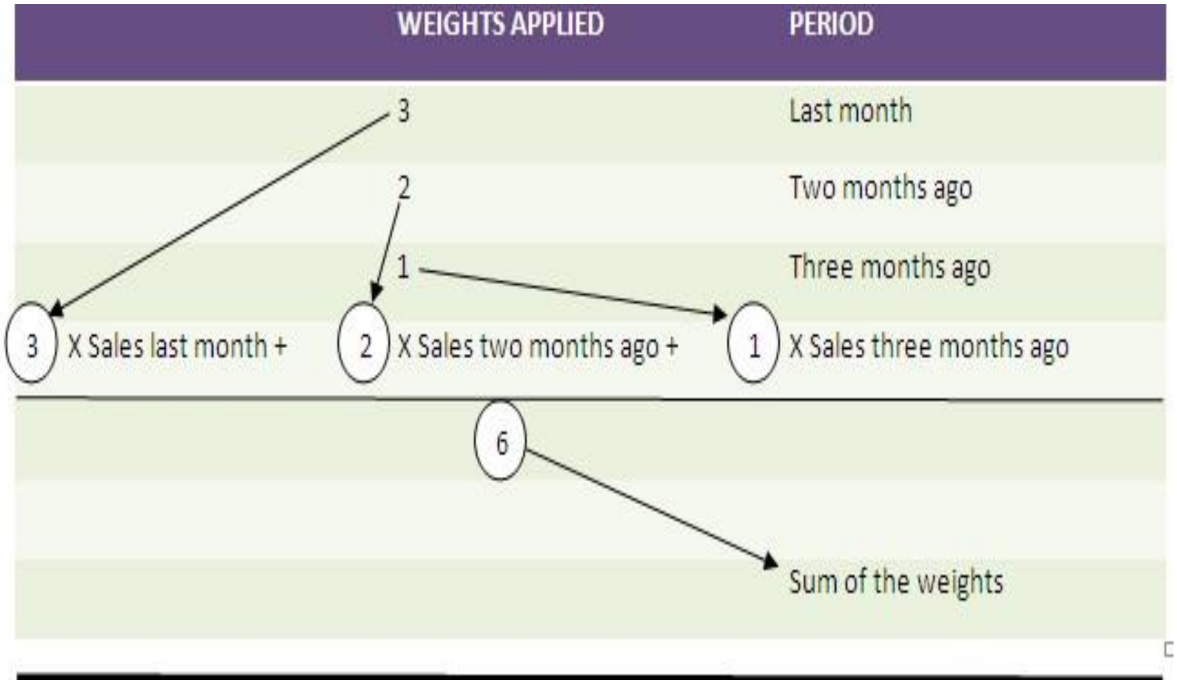
$$F_t = \frac{W_1 Y_{t-1} + W_2 Y_{t-2} + \ldots + W_n Y_{t-n}}{W_1 + W_2 + \ldots + W_n}$$

where

W_i = weight for observation in time period $t - i$

MONTH	ACTUAL SHED SALES	THREE-MONTH MOVING AVERAGE
January	10	
February	12	
March	13	
April	16	$(10 + 12 + 13)/3 = 11^2/_3$
May	19	$(12 + 13 + 16)/3 = 13^2/_3$
June	23	$(13 + 16 + 19)/3 = 16$
July	26	$(16 + 19 + 23)/3 = 19^1/_3$
August	30	$(19 + 23 + 26)/3 = 22^2/_3$
September	28	$(23 + 26 + 30)/3 = 28$
October	18	$(26 + 30 + 28)/3 = 26^1/_3$
November	16	$(30 + 28 + 18)/3 = 25^1/_3$
December	14	$(28 + 18 + 16)/3 = 20^2/_3$
January	-	$(18 + 16 + 14)/3 = 16$

Wallace Garden Supply decides to forecast storage shed sales by weighing the past three months as follows:



The results of the Wallace Garden Supply weighted average forecast are shown in Table 6.4. In this particular forecasting situation, you can see that weighting the latest month more heavily provides a much more accurate projection, and calculating the MAD for each of these would verify this.

Both simple and weighted moving averages are effective in smoothing out sudden fluctuations in the demand pattern in order to provide stable estimates. Moving averages do, however, have two problems. First, increasing the size of *n* (the number of periods averaged) does smooth our fluctuations better, but it makes the method less sensitive to real changes in the data should they occur. Second, moving averages cannot pick up trends very well. Because they are averages, they will always stay within past levels and will not predict a change to either a higher or lower level.

MONTH	ACTUAL SHED SALES	THREE-MONTH MOVING AVERAGE
January	10	
February	12	
March	13	
April	16	$[(3 \times 13) + (2 \times 12) + (10)] / 6 = 12\frac{1}{6}$
May	19	$[(3 \times 16) + (2 \times 13) + (12)] / 6 = 14\frac{1}{3}$
June	23	$[(3 \times 19) + (2 \times 16) + (13)] / 6 = 17$
July	26	$[(3 \times 23) + (2 \times 19) + (16)] / 6 = 20\frac{1}{2}$
August	30	$[(3 \times 26) + (2 \times 23) + (19)] / 6 = 23\frac{5}{6}$
September	28	$[(3 \times 30) + (2 \times 26) + (23)] / 6 = 27\frac{1}{2}$
October	18	$[(3 \times 28) + (2 \times 30) + (26)] / 6 = 28\frac{1}{3}$
November	16	$[(3 \times 18) + (2 \times 28) + (30)] / 6 = 23\frac{1}{3}$
December	14	$[(3 \times 16) + (2 \times 18) + (28)] / 6 = 18\frac{2}{3}$
January	-	$[(3 \times 14) + (2 \times 16) + (18)] / 6 = 15\frac{2}{3}$

Exponential Smoothing

Exponential smoothing is a forecasting method that is easy to use and is handled efficiently by computers. Although it is a type of moving average technique, it involves little record keeping of past data. The basic exponential smoothing formula can be shown as follows:

New forecast = last period's forecast + α (last period's actual demand – last period's forecast)

Where α is a weight (or smoothing constant) that has a value between 0 and 1, inclusive.

Equation for determining forecast using exponential smoothing can be written mathematically as

$$Ft = Ft\text{-}1 + \alpha\,(Yt\text{-}1 - Ft\text{-}1)$$

Where

Ft = new forecast (for time period t)

Ft-1 = previous forecast (for time period t - 1)

α = smoothing constant ($0 \leq \alpha \leq 1$)

Yt-1 = previous periods actual demand

The concept here is not complex. The latest estimate of demand is equal to the old estimate adjusted by a fraction of the error (last period‟s actual demand minus the old estimate).

The smoothing constant, α, can be changed to give more weight to recent data when the value is high or more weight to past data when it is low. For example, when $\alpha = 0.5$, it can be shown mathematically that the new forecast is base almost entirely on demand in the past three periods. When $\alpha = 0.1$, forecast places little weight on the single period, even the most recent, and it takes many periods (about 19) of historic values into account.

In January, a demand for 142 of a certain car model for February was predicted by a dealer. Actual February demand was 153 autos. Using a smoothing constant of $\alpha = 0.20$, we can forecast the March mean using the exponential smoothing model. Substituting into the formula, we obtain

New forecast (for March demand) = 142+ 0.2(153-142)

$$= 144.2$$

Thus, the demand forecast for the cars in March was 136. A forecast for the demand in April, using the exponential smoothing model with a constant of $\alpha = 0.20$, can be made:

New forecast (for April demand) = 144.2 + 0.2(136-144.2)

$$= 142.6 \text{ or } 143 \text{ autos}$$

Selecting the smoothing constant

The exponential smoothing approach is easy to use and has been applied successfully by banks, manufacturing companies, wholesalers, and other organizations. The appropriate value of the smoothing constant, α, however, can value for the smoothing constant, the objective is to obtain the most accurate forecast. Several values of the smoothing constant may be tried, and the one with the lowest MAD could be selected. This is analogous to how weights are selected for a weighted moving constant. QM for windows will display the MAD that would be obtained with values of α ranging from 0 to 1 increments of
0.01.

Port of Baltimore Example

Let us apply this concept with a trial-and-error testing for two values of α in the following example. The port of Baltimore has unloaded large quantities of grain from ships during the past eight quarters. The port‟s operations manger wants to test the use of exponential smoothing to see how well the technique works in predicting tonnage unloaded. He assumes that the forecast of grain unloaded in the first quarter was 175 tons. Two values of α are examined: α = 1.0 and α = .50. Table 7.5 shows the detailed calculations for the α = 0.10 only.

To evaluate the accuracy of each smoothing constant, we can compute the absolute deviations and MADs (see Table 7.6). Based on this analysis, a smoothing constant of α = 0.10 is preferred to α = 0.50 because it‟s MAD is smaller.

QUATER	ACTUAL TONNAGE UNLOADED	ROUNDED FORECAST USING α = 0.10*	ROUNDED FORECAST USING α = 0.10*
1	180	175	175
2	168	176 = 175.00 + 0.10(180 - 175)	178
3	159	175 = 175.50 + 0.10 (168 - 175.50)	173
4	175	173 = 174.75 + 0.10 (159 - 174.75)	166
5	190	173 = 173.18 + 0.10 (175 - 173.18)	170
6	205	175 = 173.36 + 0.10 (190 - 173.36)	180
7	180	178 = 175.02 + 0.10 (200 - 175.02)	193
8	182	178 = 178.02 + 0.10 (180 - 178.02)	186
9	?	179 = 178.22 + 0.10 (182 - 178.22)	184

QUARTER	ACTUAL TONAGE UNLOADED	ROUNDED FORECAST WITH 0.10	ABSOLUTE DEVIATIONS FOR 0.10	ROUNDED FORCAST WITH 0.50	ABSOLUTE DEVIATIONS FOR 0.50
1	180	175	5	175	5
2	168	176	8	178	10
3	159	175	16	173	14
4	175	173	2	166	9
5	190	173	17	170	20
6	205	175	30	180	25
7	180	178	2	193	13
8	182	178	4	186	4
		Sum of absolute deviations	84		100 MAD = 12.50

Using Excel QM for exponential Smoothing Programs 6.2A and Programs 6.2B illustrate how Excel QM handles exponential smoothing. Input data and formulas appear in Program 6.2A and output, using α of 0.1 for the port of Baltimore, are in Program 6.2B. Note that the MAD in (of 10.307) differs slightly from that in Table 6.6 because of rounding.

Decision–Making Group: A group of experts in a Delphi technique that has the responsibility of making the forecast.

Delphi: A judgmental forecasting technique that uses decision makers, staff personnel, and respondents to determine a forecast.

Error: The difference between the actual value and the forecast value.

Exponential Smoothing: A forecasting method that is a combination of the last forecast and the observed value.

Least Squares: A procedure used in trend projection and regression analysis to minimize the squared distances between the estimated straight line and the observed values.

Mean Absolute Deviation (MAD): A technique for determining the accuracy of a forecasting model by taking the average of the absolute deviations.

Mean Absolute Percent Error (MAPE): A technique for determining the accuracy of a forecasting model by taking the average of the absolute errors as percentage of the observed values.

Mean Squared Error (MSE): technique for determining the accuracy of a forecasting model by taking the average of the squared error terms for the forecast.

Moving Average: A forecasting technique that averages past values in computing the forecast.

Naïve Model: A time-series forecasting model in which the forecast for next period is the actual value for the current period.

Qualitative Models: Models that forecast using judgments, experience, and qualitative and subjective data.

Smoothing Constant: A value between 0 and 1 that is used in an exponential smoothing forecast.

Weighted Moving Average: A moving average forecasting method that places different weights on past values.

SOLVED PROBLEMS

Demand for patient surgery at Washington General Hospital has increased steadily in the past few years as seen in the following table.

YEAR	OUTPATIENT SURGERIES PERFORMED
1	45
2	50
3	52
4	56
5	58
6	-

The director of medical services predicted six years ago that demand in year 1 would be 42 surgeries. Using exponential smoothing with a weight of =0.20, develop forecasts for years 2 through 6. What is the MAD?

FINANCIAL OTABIL

YEAR	ACTUAL	FORECAST (SMOOTHED)	ERROR	|ERROR|
1	45	42	+3	3
2	50	42.6=42+0.2(45-42)	+7.4	7.4
3	52	44.1=42.6+0.2(50-42.6)	+7.9	7.9
4	56	45.7=44.1+0.2(52-44.1)	+10.3	10.3
5	58	47.7=45.7+0.2(56-45.7)	-	-
6	-	49.8=47.7+.2(58-47.7)	-	-
				38.9

PROBLEMS

1. Develop a four-month moving average forecast for Wallace Garden Supply and compute the MAD. A three-month moving average forecast was developed in the section on moving averages in Table 6.3.

2. Using MAD, determine whether the forecast in Problems 6-12 or the forecast in the section concerning Wallace Garden Supply is more accurate.

3. Data collected on the yearly demand for 50-pound bags of fertilizer at Wallace Garden Supply are shown in the following table. Develop a three-year moving average to forecast sales. Then estimate demand again with a weighted moving average in which sales in the most

recent year are given a weight of 2 and sales in the other two years are each given a weight of 1. Which method do you think is?

YEAR	DEMAND FOR FERTILIZER (1,000s of Bags)
1	4
2	6
3	4
4	5
5	10
6	8
7	7
8	9
9	12
10	14
11	15

4. Use exponential smoothing with a smoothing constant of 0.3 to forecast the demand for fertilizer given in Problem 6.14.Assume that the last period"s forecast for year 1 is 5000 bags to begin the procedure. Would you prefer to use the exponential smoothing model or the weighted average model developed on Problem 3.

5. Sales of Cool-Man air conditioners have grown steadily during the past five years.

YEAR	SALES
1	450
2	495
3	518
4	563
5	584
6	?

The Sales manager had predicted, before the business started, that year 1‟s sales would be 410 air conditioners. Using exponential smoothing with a weight of $\alpha = 0.30$, develop forecasts for years 2 through 6.

6. What effect did the smoothing constant have on the forecast for Coolman air conditioners? Which smoothing constant gives the most accurate forecast?

7. Use a three-year moving average forecasting model to forecast the sales of Cool-Man air conditioners.

8. Sales of industrial vacuum cleaners at R. Low enthal Supply Co. over the past 13 months are as follows:

SALES (1,000s)	MONTH	SALES (1,000s)	MONTH
11	January	14	August
14	February	17	September
16	March	12	October
10	April	14	November
15	May	16	December
17	June	11	January
11	July		

a. Evaluate the accuracy of each of these methods

9. Passenger miles flown on Northeast Airlines, a commuter firm serving the Boston hub, are as follows for the past 12 weeks;

WEEK	ACTUAL PASSENGER (1,000s)
1	17
2	21
3	19

4	23
5	18
6	16
7	20
8	18
9	22
10	20
11	15
12	22

10. Emergency calls to Winter Park, Florida"s 911 system, for past 24 weeks are as follows:

WEEK	CALLS	WEEK	CALLS
1	50	13	55
2	35	14	35

3	25	15	25
4	40	16	55
5	45	17	55
6	35	18	40
7	20	19	35
8	30	20	60
9	35	21	75
10	20	22	50
11	15	23	40
12	40	24	65

a. Compare the exponentially smoothed forecast of calls for each week. Assume an initial forecast of 50 calls in the first week and use α = 0.1. What is the forecast for the 25th week?

b. Reforecast each period using α = 0.6

c. Actual calls during the 25th week were 85. Which smoothing constant provides a superior forecast?

11. Using the 911 call data in Problem 9, forecast calls for weeks 2 through 25 using α = 0.9. Which is best? (Again, assume that actual calls in week 25 were 85 and use an initial forecast of 50 calls.)

12. Consulting income at Kate Walsh Association for the period February-July has been as follows:

MONTH	INCOME(1,000s)
February	70.0
March	68.5
April	64.8
May	71.7
June	71.3
July	72.8

Use exponential smoothing to forecast Augusts" income. Assume that the initial forecast for February is $65,000. The smoothing constant selected is $\alpha = 0.1$.

Resolve Problem 12 with $\alpha = 0.3$. Using MAD, which smoothing constant provides a better forecast?

CHAPTER SEVEN

MATRICES AND ITS APPLICATIONS

INTRODUCTION

A business may collect and store or analyze various types of data as a regular part of its record-keeping procedures. The data may be presented in tabular form.

For example, a store owner who sells different building materials may want to know his/her daily sales from each of the building materials. Table 1

	Iron rods	Cement	Nails
Shop A	1200	7350	700
Shop B	950	8000	980

If we write the numbers from table 1 in the rectangular array

$$P = \begin{bmatrix} 1200 & 7350 & 700 \\ 950 & 8000 & 980 \end{bmatrix}$$

We say that P is a matrix (plural: matrices) representing table 1.

In addition to storing data in a matrix, we can analyze data and make business decisions by defining the operations of addition, subtraction and multiplication for matrices.

In general, we define a **matrix** as any rectangular array of numbers/elements. The numbers in a matrix are called a matrix its **entries** or **elements**

Example of matrices is:

$$A = \begin{pmatrix} 23 \\ 12 \end{pmatrix}$$

$$B = (12 \; 11)$$

$$C = \begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix}$$

$$D = \begin{pmatrix} 2 & 1 & 5 \\ 3 & 4 & 1 \\ 3 & 5 & 6 \end{pmatrix}$$

SIZE OF A MATRIX

The size of a matrix is determined by its number of rows by the number of columns

The matrix $A = \begin{pmatrix} 23 \\ 12 \end{pmatrix}$, has 2 rows and 1 column. Therefore A is a 2×1 matrix

The matrix $B = (1 \; 3 \; 7)$, has 1 row and 3 columns. Therefore B is a 1×3 matrix

The matrix $C = \begin{pmatrix} 2 & 4 \\ 1 & 3 \end{pmatrix}$, has 2 rows and 2 columns

Therefore C is a 2×2 matrix

The matrix $D = \begin{pmatrix} 3 & 1 & 5 \\ 2 & 3 & 7 \end{pmatrix}$, has 2 rows and 3 columns

Therefore D is a 2×3 matrix.

If the number of rows of a matrix equals the number of columns, we say the matrix is a **square matrix**

ADDITION AND SUBTRACTION OF MATRICES

We can only add or subtract matrices which are of the same dimension (size)

Example 1

Given that $A = \begin{pmatrix} 3 & 5 \\ -1 & 7 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 3 \\ -3 & 4 \end{pmatrix}$, find $A + B$.

Solution

$$A + B = \begin{pmatrix} 3 & 5 \\ -1 & 7 \end{pmatrix} + \begin{pmatrix} 2 & 3 \\ -3 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 3+2 & 5+3 \\ -1+(-3) & 7+4 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & 8 \\ -4 & 11 \end{pmatrix}$$

Example 2

Given that $C = \begin{pmatrix} 2 & 5 & 3 \\ 3 & 6 & 2 \end{pmatrix}$ and $D = \begin{pmatrix} 3 & 4 & 1 \\ 1 & 2 & 2 \end{pmatrix}$, find $C - D$

Solution

$$C - D = \begin{pmatrix} 4 & 5 & 3 \\ 3 & 6 & 2 \end{pmatrix} - \begin{pmatrix} 3 & 4 & 1 \\ 1 & 2 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 4-3 & 5-4 & 3-1 \\ 3-1 & 6-2 & 2-2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 2 \\ 2 & 4 & 0 \end{pmatrix}$$

MULTIPLICATION OF MATRICES BY SCALAR

When a matrix is multiplied by a scalar k, we use the scalar to multiply each of the elements in the matrix.

If $P = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $kP = k \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$= \begin{pmatrix} ka & kb \\ \end{pmatrix}$$

□□

□*kc kd*□□

Example 3

□3 5□ □2 3□

M □□

Given that □14□□ and *N* □□□16□□, find 2*M N*□3

Solution

□3 5□ □2 3□

2*M N*□3 □2□1 4□□□3□□1 6□□

□

□6 10□ □6 9 □

□□□2 8 □ □□ □□ 3 18□□

□6 6 10 9□□ □

□□□2 3 8 18□ □ □□

□12 19□

□□ 5 26□□

□

Example 4

□2 1□ □3 2□

M □□□3 2□□ and *N* 3□, find 4*N M*□3

□□□1 □

5□□

Given that □4 2□□ □□3

□

Solution

□3 2□ □2 1□

4*N M*□3□ 4 1□□3□□□3 3□□2□□

□3 5□□ □□4 2□□

□

□12 8 □ □ 6 3□

□

□□ 412□ □□ □□ 96□□

□1220□ □□ □126□□

□

□ 12□6 8□3 □ □ 6

□

□□ 4□9 12□6□ □□ □□ □5 5 □

6 □

□12 12□ 20□6□ □□ □ 0 □

□ 14□□

Example 5

The total sales in cedis made by two shops „God Reigns‟ and „Good Father‟ on goods sold in the months of April and May is as shown below.

$$\begin{array}{c} \textbf{April} \\ \begin{array}{lcc} & \textbf{Rice} & \textbf{Cooking Oil} \\ \textbf{God Reigns} & & \\ \textbf{Good Father} & & \end{array} \end{array}$$

$$\begin{array}{l} \textbf{God Reigns} \\ \textbf{Good Father} \end{array} \begin{pmatrix} 6000 & 3200 \\ 4500 & 3200 \end{pmatrix} = A \qquad \begin{array}{l} \textbf{God Reigns} \\ \textbf{Good Father} \end{array} \begin{pmatrix} 6500 & 4200 \\ 5000 & 4000 \end{pmatrix} = B$$

(columns: **Rice**, **Cooking Oil**; left matrix **April**, right matrix **May**)

(a) What was the combined cedi sale from the months of April and May?

(b) What was the increase in the cedi sales from April to May?

(c) If both shops received 10% increase in sales in the month of June from May sales and had 5% decrease in July sales from June sales, compute the total sales from the months of June and July.

Solution

$$A = \begin{pmatrix} 6000 & 3200 \\ 4500 & 3200 \end{pmatrix}, \quad B = \begin{pmatrix} 6500 & 4200 \\ 5000 & 4000 \end{pmatrix}$$

(a) $$A + B = \begin{pmatrix} 6000 & 3200 \\ 4500 & 3200 \end{pmatrix} + \begin{pmatrix} 6500 & 4200 \\ 5000 & 4000 \end{pmatrix}$$

$$= \begin{pmatrix} 6000+6500 & 3200+4200 \\ 4500+5000 & 3200+4000 \end{pmatrix}$$

$$= \begin{pmatrix} 12500 & 7400 \\ 9500 & 7200 \end{pmatrix}$$

(b) $$B - A = \begin{pmatrix} 6500 & 4200 \\ 5000 & 4000 \end{pmatrix} - \begin{pmatrix} 6000 & 3200 \\ 4500 & 3200 \end{pmatrix}$$

$$= \begin{pmatrix} 6500-6000 & 4200-3200 \\ \end{pmatrix}$$

5000 4500 4000 3200

500 1000

500 800

6500 4200

(c) June : C 1.1 5000 4000

7150 4620

5500 4400

7150 4620

July: D 0.95 5500 4400

6792.5 4389

5225 4180

7150 4620 6792.5 4389

C D 5500 4400 5225 4180

13942 9009

10725 8580

MULTIPLICATION OF MATRICES

Two matrices can be multiplied if the number of rows of the first matrix is equal to the number of columns of the second matrix.

If matrix A has size $m \times n$ and matrix B has size $n \times p$, then the matrix AB can be multiplied with size $m \times p$

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$, then

$$AB = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix}$$

$$= \begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix}$$

Example 6

Given that $P = \begin{pmatrix} 1 & -2 \\ -3 & 2 \end{pmatrix}$, $Q = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$, $R = (3\ 5)$ and $S = \begin{pmatrix} -2 & 3 \\ \frac{1}{4} & 1 \end{pmatrix}$, find

(i) PQ (ii) PR (iii) PS (iv) QR (v) QS (vi) RS

Solution

(i) $P_{(2 \times 2)} Q_{(2 \times 1)} = \begin{pmatrix} 1 & -2 \\ -3 & 2 \end{pmatrix}\begin{pmatrix} -3 \\ 2 \end{pmatrix}$

$$= \begin{pmatrix} -3 - 4 \\ 9 + 4 \end{pmatrix}$$

$$= \begin{pmatrix} -7 \\ 13 \end{pmatrix}$$

$\begin{pmatrix} 1 & -2 \end{pmatrix}$

(ii) (2 2) (1 2)P R□ □ □□□3 2□□(3 5) [Multiplication not possible]

□12□□2 1 3□
4

(iii) P S □□3
(2 2)□ (2 3)□

2□□□□2 1□□
□

□2□ 4 1□8 3□ 2□

□□6□ 4 3□8 9□ 2□□
□

□ 6 9 5 □

□□□10 11 11□□

□ □3

(iv) Q R□ □ □□ □2
(3 5)

(2 1) (1 2) □ □

□9 15□

□□6 10□□
□

□ □□32 1 3□

(iv) (2 1)Q S□ (2 3)□ □□ □□□ □□2 2 4 1□□ [Multiplication not possible]

□2 1 3□

(vi) (1 2)R S□ (2 3)□ □ (3 5)□□24 1□□

□ (6□10 3□ 20 9□5)

□ (16 23 14)

DETERMINANT OF A MATRIX

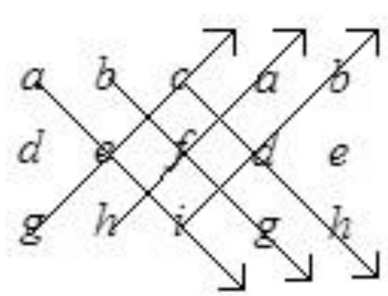
If $A_{2\times 2} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then det A or $|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

If $C_{3\times 3} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ then

$$\det C = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

OR we can also find the determinant of a 3×3 matrix by

1. First writing the matrix without the bracket
2. Adjoin the matrix with the first two columns of the same matrix and find the determinant as shown below.



Determinant = sum of the product of downward arrows – sum of the product of the upward arrows.

Example 7

If $A = \begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix}$, find the determinant of A

Solution

$$|A| = (3 \times 2) - (2 \times 1)$$
$$= 6 - 2$$
$$= 4$$

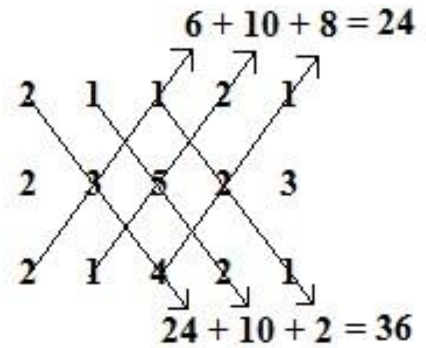
Example 8

If $P = \begin{pmatrix} 2 & 1 & 1 \\ 2 & 3 & 5 \\ 2 & 1 & 4 \end{pmatrix}$, find the determinant of P

Solution

$$|P| = 2\begin{vmatrix} 3 & 5 \\ 1 & 4 \end{vmatrix} - 1\begin{vmatrix} 2 & 5 \\ 2 & 4 \end{vmatrix} + 1\begin{vmatrix} 2 & 3 \\ 2 & 1 \end{vmatrix}$$
$$= 2(12 - 5) - 1(8 - 10) + 1(2 - 6)$$
$$= 2(7) - 1(-2) + 1(-4)$$
$$= 14 + 2 - 4$$
$$= 12$$

ALTERNATIVELY



$$|P| = 36 - 24$$
$$= 12$$

SOLUTION OF LINEAR SIMULTANEOUS EQUATIONS USING CRAMER'S RULE

1. TWO LINEAR SIMULTANEOUS EQUATIONS

To solve the system of equations

$$ax + by = e$$
$$cx + dy = f$$

We first rewrite the equations as

$$\begin{pmatrix} ax + by \\ cx + dy \end{pmatrix} = \begin{pmatrix} e \\ f \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} e \\ f \end{pmatrix}$$

Represent the left hand matrix by a letter say $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and the right hand side matrix by say $B = \begin{pmatrix} e \\ f \end{pmatrix}$

To find x, replace the first column of A by B, find its determinant and divide the results by the determinant of A

Thus: $x = \dfrac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{|A|}$

To find y, replace the second column of A by B, find its determinant and divide the results by the determinant of A

Thus: $y = \dfrac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{|A|}$

Example 9

Solve the system of equations $\begin{matrix} 2x + 3y = 23 \\ x + 2y = 14 \end{matrix}$,using Cramer‟s rule

Solution

$2x + 3y = 23$

$x + 2y = 14$

$\Rightarrow \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 23 \\ 14 \end{pmatrix}$

Let $A = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 23 \\ 14 \end{pmatrix}$

$|A| = 2 \times 2 - 1 \times 3$

$= 1$

$$x = \frac{\begin{vmatrix} 23 & 3 \\ 14 & 2 \end{vmatrix}}{1} = \frac{23(2) - 14(3)}{1} = \frac{46 - 42}{1} = 4$$

$$y = \frac{\begin{vmatrix} 2 & 23 \\ 1 & 14 \end{vmatrix}}{1} = \frac{14(2) - 1(23)}{1} = \frac{28 - 23}{1} = 5$$

$\therefore x = 4$

$y = 5$

Example 10

Pascal‟s logistics company has an order for two products to be delivered to two stores. The matrix below gives information regarding the two products

	Product I	Product II
Unit volume (cu ft)	20	30
Unit weight (lb)	100	400

If a truck can carry 2350 cu ft and 23,000 lb, how many of each product can it carry.

Solution

Let x = the number of product I it can carry

y = the number of product II it can carry

$$20x + 30y = 2350$$
$$100x + 400y = 23000$$

$$\begin{pmatrix} 20 & 30 \\ 100 & 400 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2350 \\ 23000 \end{pmatrix}$$

Let $A = \begin{bmatrix} 20 & 30 \\ 100 & 400 \end{bmatrix}$, $B = \begin{bmatrix} 2350 \\ 23000 \end{bmatrix}$

$$|A| = 20 \times 400 - 100 \times 30$$
$$= 5000$$

$$x = \frac{\begin{vmatrix} 2350 & 30 \\ 23000 & 400 \end{vmatrix}}{5000} = \frac{2350 \times 400 - 23000 \times 30}{5000}$$
$$= \frac{940,000 - 690,000}{5000}$$
$$= 50$$

$$y = \frac{\begin{vmatrix} 20 & 2350 \\ 100 & 23000 \end{vmatrix}}{500} = \frac{20 \times 23000 - 100 \times 2350}{500}$$
$$= \frac{460,000 - 235,000}{500}$$
$$= 450$$

$\therefore x = 50$

$y = 450$

2. THREE LINEAR SIMULTANEOUS EQUATIONS

To solve the system of equations

$$ax + by + cz = p$$
$$dx + ey + fz = q$$
$$gx + hy + ix = r$$

We first rewrite the equations as

$$\begin{cases} ax + by + cz = p \\ dx + ey + fz = q \\ gx + hy + iz = r \end{cases}$$

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} p \\ q \\ r \end{pmatrix}$$

Represent the left hand matrix by a letter say $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ and the right hand side matrix by say $B = \begin{pmatrix} p \\ q \\ r \end{pmatrix}$

To find x, replace the first column of A by B, find its determinant and divide the results by the determinant of A

Thus: $x = \dfrac{\begin{vmatrix} p & b & c \\ q & e & f \\ r & h & i \end{vmatrix}}{|A|}$

To find , replace the second column of y A by , find its determinant$B$ and divide the results by the determinant of A

Thus: y □ $\dfrac{\begin{vmatrix} a & p & c \\ d & q & f \\ g & r & i \end{vmatrix}}{|A|}$

To find , replace the third column of bz$A$ y , find its determinant and divide the Bresults by the determinant of A

Thus: z □ $\dfrac{\begin{vmatrix} a & b & p \\ d & e & q \\ g & h & r \end{vmatrix}}{|A|}$

Example 8

$$X_1 \square 3X_2 \square\ 2X_3 \square 14$$

$$5X_1 \square\ 2X_2 \square\ X_3 \square 13$$

Solve the system of equations $2X_1 \square\ X_2 \square\ 4X_3 \square 13$,using Cramer‟s rule

Solution

$X_1 \square 3X_2 \square\ 2X_3 \square 14$

$5X_1 \square\ 2X_2 \square\ X_3 \square 13$ $2X_1 \square$

$X_2 \square\ 4X_3 \square 13$

□1 3 2□□ X_1 □ □14□

□□5 2 1□□□□ X_2 □□

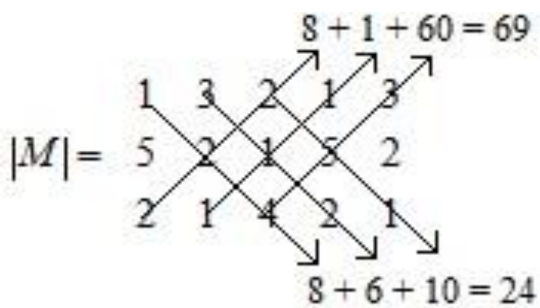
□ □□13□□ □□2 1

4□□□□ X_3 □□ □□13□□

□1 3 2□ □14□

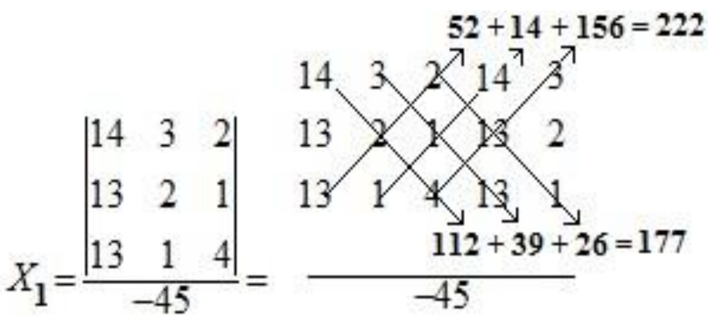
Let M □ □□5 2 1□□,N □ □□13□□

□□2 1 4□□ □□13□□



□ 24□69

□ □45



222 □177 □

□45

□45

□ □ 1

□45

$$X_2 = \frac{\begin{vmatrix} 1 & 14 & 2 \\ 5 & 13 & 1 \\ 2 & 13 & 4 \end{vmatrix}}{-45} = \frac{\begin{array}{c} 52+13+280=345 \\ \begin{array}{ccccc} 1 & 14 & 2 & 1 & 14 \\ 5 & 13 & 1 & 5 & 13 \\ 2 & 13 & 4 & 2 & 13 \end{array} \\ 52+28+130=210 \end{array}}{-45}$$

$$= \frac{345 - 210}{-45}$$

$$= \frac{-135}{-45} = 3$$

$$X_3 = \frac{\begin{vmatrix} 1 & 3 & 14 \\ 5 & 2 & 13 \\ 2 & 1 & 13 \end{vmatrix}}{-45} = \frac{\begin{array}{c} 56+13+195=264 \\ \begin{array}{ccccc} 1 & 3 & 14 & 1 & 3 \\ 5 & 2 & 13 & 5 & 2 \\ 2 & 1 & 13 & 2 & 1 \end{array} \\ 26+78+70=174 \end{array}}{-45}$$

$$= \frac{264 - 174}{-45}$$

$$= \frac{-90}{-45} = 2$$

$\therefore X_1 = 1, X_2 = 3, X_3 = 2$

Example 10

Walters Manufacturing Company needs to know how best to use the time available within its three manufacturing departments in the construction and packaging of the three types of metal storage sheds. Each one must be stamped, painted and packaged. The table below shows the number of hours required for the processing of each type of shed.

	Shed		
Department	**Type I**	**Type II**	**Type III**
Stamping	2	3	4
Painting	1	2	1
Packaging	1	1	2

Determine how many of each type of shed can be produced if the stamping department has 3200 hours available, the painting department has 1700 hours, and the packaging department has 1300 hours.

Solution

Let X be the number of type I sheds, Y be the number of type II sheds and Z be the number of type III sheds.

The equation $2X + 3Y + 4Z = 3200$ represents the hours used by the stamping department.

Similarly; $1X + 2Y + 1Z = 1700$ represents the hours used by the painting department and $1X + 1Y + 2Z = 1300$ represents the hours used by the stamping department

2*X Y Z*□3 □ 4 □ 3200

1*X Y Z*□ 2 □1 □1700 1*X*

Y Z□1 □ 2 □1300

□2 3 4□□ □ □*X* 3200□

□1 21□□ □ □□□ □ □*Y* □ 1700 □□

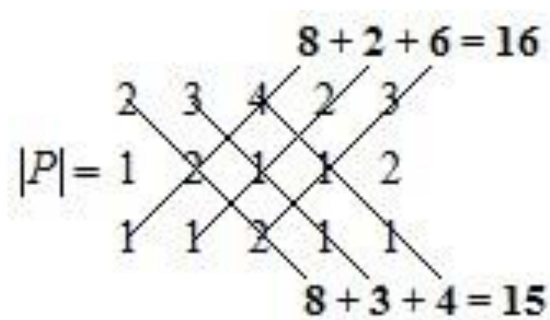
□

□□1 12□□ □ □□□ □ □*Z* 1300 □□

□2 3 4□ □3200□

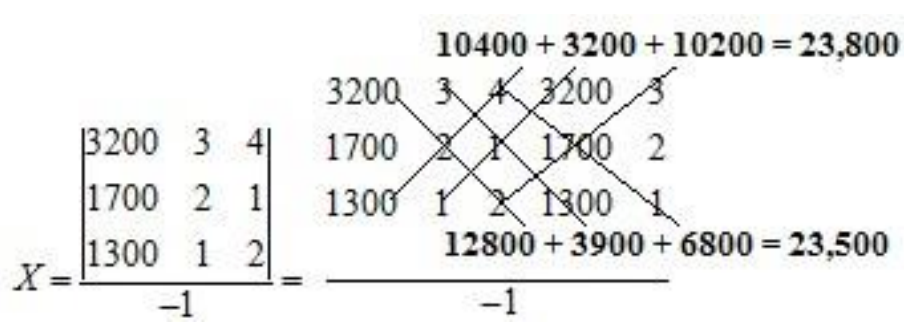
Let *P*□ □ □1 2 1 □ □, *Q*□ □ □1700 □ □

□□1 1 2□□ □□1300 □□



□15 16□

□ □1



23800 23500□

□

□1

$$= \frac{-300}{-1}$$

$$= 300$$

$$Y = \frac{\begin{vmatrix} 2 & 3200 & 4 \\ 1 & 1700 & 1 \\ 1 & 1300 & 2 \end{vmatrix}}{-1} = \frac{\overset{6800 + 2600 + 6400 = 15{,}800}{\underset{6800 + 3200 + 5200 = 15{,}200}{\begin{matrix} 2 & 3200 & 4 & 2 & 3200 \\ 1 & 1700 & 1 & 1 & 1700 \\ 1 & 1300 & 2 & 1 & 1300 \end{matrix}}}}{-1}$$

$$= \frac{15800 - 15200}{-1}$$

$$= \frac{-600}{-1}$$

$$= 600$$

$$Z = \frac{\begin{vmatrix} 2 & 3 & 3200 \\ 1 & 2 & 1700 \\ 1 & 1 & 1300 \end{vmatrix}}{-1} = \frac{\overset{6400 + 3400 + 3900 = 13{,}700}{\underset{5200 + 5100 + 3200 = 13{,}500}{\begin{matrix} 2 & 3 & 3200 & 2 & 3 \\ 1 & 2 & 1700 & 1 & 2 \\ 1 & 1 & 1300 & 1 & 1 \end{matrix}}}}{-1}$$

$$= \frac{13700 - 13500}{-1}$$

$$= \frac{-200}{-1}$$

$$= 200$$

Therefore, the company should make 300 type I, 600 type II and 200 type III sheds

EXERCISES

1. Given that $A = \begin{pmatrix} 3 & 5 \\ 1 & 7 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix}$ and $C = \begin{pmatrix} 5 & 7 \\ 4 & 3 \end{pmatrix}$ find

   (i) $A \; B$

   (ii) $A \; C \; B$

   (iii) $B \; C \; A$

2. Given that $C = \begin{pmatrix} 3 & 3 & 2 \\ 1 & 4 & 2 \\ 3 & 1 & 1 \end{pmatrix}$ and $D = \begin{pmatrix} 4 & 7 & 3 \\ 5 & 6 & 9 \\ 4 & 3 & 3 \end{pmatrix}$, find

   (i) $C \; D$

   (ii) $D \; C$

3. Given that $A = \begin{pmatrix} 3 & 3 & 5 \\ 1 & 4 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 4 & 3 \\ 3 & 2 & 4 \end{pmatrix}$ and $C = \begin{pmatrix} 1 & 3 & 2 \\ 2 & 5 & 3 \end{pmatrix}$ find

   (i) $3A \; B$

   (ii) $A \; B \; C \; 2$

   (iii) $3C \; A \; B$

   (iv) $2B \; C \; A$

4. The total number of furniture made by two shops „Lom Na Va" and „Saviour" in the months of July and August is as shown below.

$$\begin{array}{ccc} & \text{July} & \\ & \text{Tables} & \text{Chairs} \\ \text{Lom Na Va} & \multicolumn{2}{c}{\multirow{2}{*}{$\begin{pmatrix} 700 & 850 \\ 650 & 990 \end{pmatrix} = A$}} \\ \text{Saviour} & & \end{array}$$

$$\begin{array}{r} \\ \text{Lom Na Va} \\ \text{Saviour} \end{array} \begin{array}{c} \text{July} \\ \begin{array}{cc}\text{Tables} & \text{Chairs}\end{array} \\ \begin{pmatrix} 700 & 850 \\ 650 & 990 \end{pmatrix} = A \end{array} \qquad \begin{array}{r} \\ \text{Lom Na Va} \\ \text{Saviour} \end{array} \begin{array}{c} \text{August} \\ \begin{array}{cc}\text{Tables} & \text{Chairs}\end{array} \\ \begin{pmatrix} 800 & 850 \\ 700 & 1100 \end{pmatrix} = B \end{array}$$

(a) What was the combined furniture made from the months of July and August?

(b) What was the increase in the cedi sales from April to May?

(c) If both shops had 15% increases in production in the month of September from August production and had 10% increase in October from July production, compute the total production from the months of September and October.

□3 4□ □ □4 □3 2 1□

5.Given that *A*□□21□□, *B*□□ □□ □1 , *C* □ (2 7) and *D*□□□5 3 6□□, find

□

(a) *AB* (b) *BA* (c) *AC* (d) *CA* (e) *AD* (f) *DA*

(g) *BC* (h) *CB* (i) *BD* (j) *DB* (k) *CD*

6. Find the determinant of each of the following matrices

□3 1 1□ □5 2 3□

(iii) *C* □□ □

(i) *A*□□□3243□□□ (ii) *B*□□□□43 12□□□□□□12 2334□□□ (iv) *D*□□□□□12

43 43□□□□

□

7. Solve each of the following system of equations, using the Cramer‟s rule

(i)
$x\ y□\ □\ 7$
$3x□\ □2y\ 17$

(ii)
$5x□\ □3y\ 7$
$6x□\ □4y\ 9$

(iii)
$2X\ Y\ Z□\ □\ □\ 7$
$X\ Y\ Z□\ □\ □3\quad 2\qquad 9$
$3X\ Y\ Z□\ □\ □313$

(iv)
$3X_1□\ 2X_2□5X_3□\ 38$
$2X\ X_1□\ □_2\qquad 3X_3□\ 23$
$5X_1□3X\ X_2□\ □_3\ 26$

	Product I	Product II
Unit volume (cu ft)	20	30
Unit weight (lb)	100	400

8. If a truck can carry 2500 cu ft and 24,500 lb, how many of each product can it carry.

9. Pascal"s trucking company has an order three products, *A*, *B* and *C*, for delivery. The table gives the volume in cubic feet, the weight in pounds, and the value for insurance in cedis for a unit of each of the products.

	Product A	Product B	Product C
Unit volume (cubic feet)	24	20	40
Weight (pounds)	40	30	60
Value (cedis)	150	180	200

If the carrier can carry 8000 cubic feet and 12,400 pounds and is insured for GH¢52,600, how many of each product can be carried?